Non-Linear Effects in Quantum Electrodynamics

by

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Declaration

By Submitting this thesis electronically, I declare that the entirety of the work contained therein is my own, original work, that I am the owner of the copyright thereof and that I have not previously in its entirety or in part submitted it for obtaining any qualification.

Date: 26 November 2010

Abstract

The Euler-Heisenberg Lagrangian is used to derive equations for the electric and magnetic fields ($\vec{\mathcal{E}}(\vec{x})$ and $\vec{\mathcal{B}}(\vec{x})$ respectively) induced by the interaction of external quasistatic electric/magnetic fields and the the fields produced by classical charges and currents.

$$\vec{\mathcal{E}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y \cdot \left\{8\mathcal{F}_M\vec{D}_M + \frac{14}{c}\mathcal{G}_M\vec{H}_M\right\}$$

$$\vec{\mathcal{B}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2 c}\vec{\nabla}_x \times \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y \times \left\{-\frac{8}{c}\mathcal{F}_M\vec{H}_M + 14\mathcal{G}_M\vec{D}_M\right\}$$

In particular, the cases of the uniformly charged spherical shell in the presence of a external magnetic field and of the spherical magnetic dipole in the presence of an external electric field were investigated. It was found that the external magnetic field induced a magnetic dipole moment ($\vec{m}$) in the uniformly charged shell.

$$\vec{m} = \frac{c^2\zeta}{6\pi\epsilon_0^2}\frac{Q^2}{R}\vec{B}_0$$

The external electric field had a similar effect on the spherical magnetic dipole where it induced an electric dipole moment ($\vec{p}_\psi$).

$$\vec{p}_\psi = \frac{\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\left[36\frac{\vec{E}_0}{E_0} - 49\frac{(\vec{E}_0\cdot\hat{e}_x)\hat{e}_x}{E_0}\right]$$

These results are quite surprising since they predict effects which were not expected. Some experiments to observe these induced fields will be discussed briefly.

Opsomming

Die Euler-Heisenberg Lagrangian word gebruik om vergelykings vir die elektriese ($\vec{\mathcal{E}}(\vec{x})$) en magnetiese ($\vec{\mathcal{B}}(\vec{x})$) velde af te lei. Hierdie velde onstaan weens die interaksie tussen eksterne elektriese/magnetiese velde en die velde van klassieke ladings en strome.

$$\vec{\mathcal{E}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y \cdot \left\{8\mathcal{F}_M\vec{D}_M + \frac{14}{c}\mathcal{G}_M\vec{H}_M\right\}$$

$$\vec{\mathcal{B}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2 c}\vec{\nabla}_x \times \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y \times \left\{-\frac{8}{c}\mathcal{F}_M\vec{H}_M + 14\mathcal{G}_M\vec{D}_M\right\}$$

Ons ondersoek die geval waar 'n uniform gelaaide sferiese skil teenwoordig is in 'n eksterne magnetiese veld en waar 'n sferiese magnetiese dipool ($\vec{m}$) in 'n eksterne elektriese veld teenwoordig is. Ons vind dat die eksterne elektriese veld 'n magnetiese dipoolmoment in die uniform gelaaide skil geïnduseer.

$$\vec{m} = \frac{c^2\zeta}{6\pi\epsilon_0^2}\frac{Q^2}{R}\vec{B}_0$$

Die eksterne elektriese veld het 'n soortgelyk effek gehad op die sferiese magnetiese dipool waar dit 'n elektriese dipool ($\vec{p}_\psi$) veroorsaak het.

$$\vec{p}_\psi = \frac{\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\left[36\frac{\vec{E}_0}{E_0} - 49\frac{(\vec{E}_0\cdot\hat{e}_x)\hat{e}_x}{E_0}\right]$$

Hierdie resultate is nogal verbasend aangesien hulle gevolge voorspel wat nie verwag was nie. Sekere eksperimente wat hierdie geïnduseerde velde waarneem is kortliks bespreek.

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Chapter 1

Introduction

1.1 Classical Electrodynamics

Light is one of the most fascinating subjects in physics. It has been studied and contemplated since the earliest days. However, mankind's understanding of the nature of light has been greatly increased during the last few centuries. James Maxwell study of light in the 1860's and 1870's led to the general acceptance of the idea that light is a form of electromagnetic radiation.

During his studies, Maxwell was able to compile a list of equations to calculate electric and magnetic fields. This list consisted of four equations which became known collectively as Maxwell's equations [7]:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0}\rho \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}$$

These equations form the basis for classical electrodynamics.

Electrodynamics, as the name suggests, deals with electric and magnetic fields which change with time. This is easily seen in the equations above where $\vec{\nabla} \times \vec{E}$ and $\vec{\nabla} \times \vec{B}$ depend on $\frac{\partial \vec{B}}{\partial t}$ and $\frac{\partial \vec{E}}{\partial t}$ respectively; or where the time dependence is contained in the source terms ρ and $\vec{J}$. Maxwell's equations take on a slightly different form when they describe electric and magnetic fields within matter:

$$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t}$$

The above equations make use of the electric displacement field $\vec{D}$ and the magnetization field $\vec{H}$. These two fields give information regarding the polarization and magnetization of the medium in which the electric and magnetic fields are found. In general, it is possible to define $\vec{D}$ and $\vec{H}$ in terms of the electric and magnetic polarization vectors, $\vec{P}$ and $\vec{M}$ respectively:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \qquad\qquad \vec{H} = \frac{1}{\mu_0}\vec{B} - \vec{M}$$

According to classical electrodynamics, the vacuum does not consist of any particles. This automatically implies a linear relation between $\vec{D}$ and $\vec{E}$ as well as a linear relation between $\vec{H}$ and $\vec{B}$ since the lack of charged particles means that the medium, in this case the vacuum, will have no polarization.

1.2 Quantum Electrodynamics

The theory of quantum electrodynamics (QED) was developed using quantum field theory and the theory of special relativity to describe the electric and magnetic interactions between fundamental particles. Contributions to QED were made by Richard Feynman, Julian Schwinger and Sin-Itiro Tomonaga for which they were awarded the 1965 Nobel Prize in Physics.

This electromagnetic interaction is expressed as an exchange of photons between the particles. Where the total interaction is the combination of all the possible exchanges between the particles. Examples of the types of possible interactions include:

- the exchange of one or more photons
- an emitted photon splitting into a particle and its anti-particle before recombining into a photon

A graphical way of expressing this interaction was developed by Feynman. This graphical representation became know as Feynman Diagrams where each diagram represents a term in the perturbation expansion of the lagrangian density.

In particular, the one-loop contribution to the vacuum polarization is of special note. The one-loop contribution to the vacuum polarization corresponds to the case where an emitted photon splits into a particle and its anti-particle; which then recombines into a photon. The associated Feynman Diagram can be seen in Fig. (1.1). This particular interaction is of great interest when investigating the non-linear effects in QED.

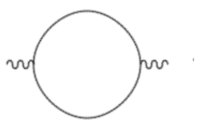

Figure 1.1: Diagrammatic representation of the one-loop contribution to the vacuum polarization

1.3 Non-Linear Quantum Electrodynamics

Non-linear processes in QED have not been investigated in great detail in the past. The effects of these processes are much smaller than the effects produced by linear QED and thus have been difficult to detect and measure. With the advent of modern lasers, with increasing field intensity and peak electric field strength of the order $10^{14}V/m$ (with $10^{15} - 10^{16}V/m$ on the horizon), the critical field strength, $E_c \approx 10^{18}V/m$, may not be so far off. Many non-linear QED effects may be detectable as electric field strengths approach this critical value. Examples of these effects include :

- Spontaneous Pair Production, which refers to the spontaneous emission of an electron-positron pair from the vacuum
- Non-linear Compton Scattering, where an electron absorbs multiple photons before emitting one

Some non-linear effects such as (i) the induced magnetic field produced by an electric charge in a constant background magnetic field and (ii) the induced electric field produced by a magnetic moment in a constant background electric field, may be detectable with present technology. These effects are a consequence of the Lagrangian proposed by W. Heisenberg and H. Euler in their paper "Consequences of Dirac's Theory of the Positron" [12].

The study of non-linear processes may lead to new insight and a better understanding of the physical universe. This study hopes to renew interest in non-linear QED topics and to introduce some novel effects generated by the interaction of the electric and magnetic fields. With further research and some proposed experiments these effects may be observable with current technologies.

1.4 Euler-Heisenberg Lagrangian

Euler and Heisenberg formulated their Lagrangian by investigating the idea that electromagnetic fields can polarize the vacuum and how this polarization affects Maxwell's equations. This polarization can be caused either by the spontaneous emission of a particle and anti-particle pair, provided that the electromagnetic fields are strong enough, or by virtually created particle and anti-particle pairs in the vacuum. The lagrangian can derived from the one-loop effective action in the presence of a background electromagnetic field. This effective action is given by [5]:

$$S^{(1)} = -i \ln \det \left(i\not{D} - m \right) = -\frac{i}{2} \ln \det \left(\not{D}^2 + \mathrm{m}^2 \right) \tag{1.1}$$

where,

$\not{D} = \gamma^{\mu}(\partial_\mu + ie\mathrm{A}_\mu)$	– Dirac operator
γ^{μ}	– gamma matrices usually represented by four 4x4 matrices
∂_μ	– derivative with respect to the space-time coordinate x^{μ}
e	– charge on an electron
m	– mass of an electron
A_μ	– fixed classical gauge potential with field strength tensor $F^{\mu\nu} = \partial_\mu \mathrm{A}_\nu - \partial_\nu \mathrm{A}_\mu$

For spinor QED, the one-loop effective action can be perturbatively expanded in even powers of the external photon field A_μ. This can be represented diagrammatically as seen in Figure 1.2. Heisenberg and Euler were able to produce

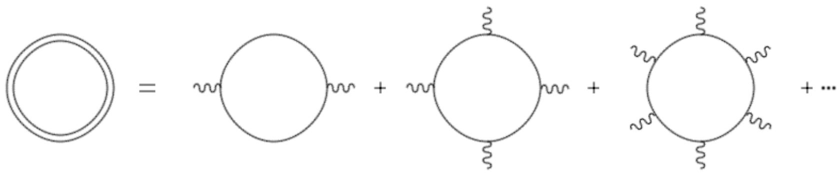

Figure 1.2: Diagrammatic representation of the one-loop effective action in the presence of a background electromagnetic field

an expression for the one-loop effective action in the low energy limit of the external photon lines (the wavy lines in Figure 1.2). Moving from the action (eqn. (1.1)) to the lagrangian requires the addition of a proper time coordinate as well as the use of the relation between the determinant (det) and the trace (tr):

$$\det(A) = \exp(\mathrm{tr}(\log(A)))$$

where A is a matrix, exp is the matrix exponential and log is the matrix logarithm. After a lot of work, Euler and Heisenberg were able to produce an expression for the lagrangian from the one-loop effective action (see [12] and [11]). This lagrangian became know as the Euler-Heisenberg Lagrangian and can be expressed as:

$$\mathcal{L}_{\mathrm{sp}}^{(1)} = \frac{1}{hc}\int_0^\infty \frac{\mathrm{d}\eta}{\eta^3}\mathrm{e}^{-\eta c\epsilon_c}\left\{\frac{e^2\mathrm{ab}\eta^2}{\tanh(cb\eta)\tan(ca\eta)} - 1 - \frac{e^2\eta^2}{3}(b^2 - a^2)\right\} \tag{1.2}$$

where,

$$\epsilon_c = \frac{m^2c^3}{e\hbar} \qquad \text{– critical field strength}$$

$$a^2 - b^2 = \vec{E}^2 - c^2\vec{B}^2 = -\frac{1}{2}F_{\mu\nu}F^{\mu\nu} \equiv 2\mathcal{F}$$

$$ab = c\vec{E}\cdot\vec{B} = -\frac{1}{4}F_{\mu\nu}\tilde{F}^{\mu\nu} \equiv \mathcal{G}$$

The critical field strength, ϵ_c, is determined as the the field strength required to produce an electron-positron pair out of the vacuum at a distance equal to that of the Compton wavelength of the electron. This is achieved by writing the energy required to produce the electron-positron pair in two different ways i.e. the energy required to create an electron-positron pair ($2mc^2$) and the energy required to move an electron and a positron, a distance equal to their Compton wavelength ($2e\epsilon_c\left(\frac{\hbar}{mc}\right)$). Equating these energies gives a value for the critical field strength:

$$\epsilon_c = \frac{m^2c^3}{e\hbar}$$

The weak field expansion of Euler-Heisenberg Lagrangian is given by:

$$\begin{aligned}
\mathcal{L}^{(1)}_{spinor} &= \frac{2\alpha^2\epsilon_0^2\hbar}{45m^4c^5}\left[(a^2-b^2)^2+7(ab)^2\right]+\cdots \\
&\approx \frac{2\alpha^2\epsilon_0^2\hbar}{45m^4c^5}\left[4\mathcal{F}^2+7\mathcal{G}^2\right]+\cdots \\
&= \zeta\left[4\mathcal{F}^2+7\mathcal{G}^2\right]+O(\zeta^2) \qquad (1.3)
\end{aligned}$$

where $\alpha = \frac{e^2}{4\pi\epsilon_0\hbar}$ is the fine structure constant and ζ is defined as:

$$\begin{aligned}
\zeta &= \frac{2\alpha^2\epsilon_0^2\hbar}{45m^4c^5} \\
&\approx 1.3\times 10^{-52}\,\frac{\text{J m}}{\text{V}^4} \qquad (1.4)
\end{aligned}$$

The first term in this expansion corresponds to the second diagram in Figure 1.2. The process is know as Light-Light Scattering and is the first non-linear effect proposed by the Euler-Heisenberg Lagrangian.

Since photons have no electric charge, produce no magnetic fields and have no mass, they cannot interact with each other. Photons are also the "information carriers" of the electromagnetic fields. This leads to the linear superposition principle of electromagnetic fields which states that the interaction between any two charges is not affected by the presence of other charges [8].

The Light-Light Scattering term says that this is not entirely true. It implies that an electric/magnetic field strength at a given point may be influenced by the background electromagnetic fields. These corrections to the electric/magnetic fields are very small compared to the electric/magnetic fields and may not be detectable. However, the induced magnetic/electric fields, which were not present before, may be detectable.

This investigation deals with these induced electric/magnetic fields i.e. the induced magnetic field produced by an electric charge in a constant background magnetic field and the induced electric field produced by a magnetic dipole in a constant background electric field.

Chapter 2

Theory

2.1 Derivation of the Induced Electric and Magnetic Field Equations

2.1.1 Defining the Auxiliary Fields $\vec{D}$ and $\vec{H}$

Our goal is to derive time-independent equations for the induced electric field and the induced magnetic field produced by an arbitrary charge (j_0) and current ($\vec{j}$). Using eqn. (1.3), we define the Lagrangian $\mathcal{L}$ as:

$$\begin{aligned}\mathcal{L} &= \epsilon_0 \mathcal{F} + \mathcal{L}^{(1)}_{spinor} \\ &= \epsilon_0 \mathcal{F} + \zeta \left[4\mathcal{F}^2 + 7\mathcal{G}^2\right] \end{aligned} \tag{2.1}$$

We define the electric displacement ($\vec{D}$) and the magnetization field ($\vec{H}$) as [4,12]:

$$\vec{D} = \frac{\partial \mathcal{L}}{\partial \vec{E}} \tag{2.2}$$

$$\vec{H} = -\frac{\partial \mathcal{L}}{\partial \vec{B}} \tag{2.3}$$

Using the chain-rule for differentiation $\frac{\mathrm{d}}{\mathrm{d}x} f(u) = \frac{\mathrm{d}f}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}$,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \vec{E}} &= \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\frac{\partial \mathcal{F}}{\partial \vec{E}} + \frac{\partial \mathcal{L}}{\partial \mathcal{G}}\frac{\partial \mathcal{G}}{\partial \vec{E}} \\ &= \left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)\vec{E} + \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)c\vec{B}\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial \vec{B}} &= \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\frac{\partial \mathcal{F}}{\partial \vec{B}} + \frac{\partial \mathcal{L}}{\partial \mathcal{G}}\frac{\partial \mathcal{G}}{\partial \vec{B}} \\ &= -\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)c^2\vec{B} + \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)c\vec{E}\end{aligned}$$

Thus,

$$\vec{D} = \left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)\vec{E} + \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)c\vec{B} \tag{2.4}$$

$$\vec{H} = \left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right) c^2 \vec{B} - \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right) c\vec{E} \tag{2.5}$$

Putting in the explicit values for $\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = (\epsilon_0 + 8\zeta\mathcal{F})$ and $\frac{\partial \mathcal{L}}{\partial \mathcal{G}} = 14\zeta\mathcal{G}$ from eqn. (2.1), we obtain expressions for $\vec{D}$ and $\vec{H}$.

$$\begin{aligned}\vec{D} &= \epsilon_0 \vec{E} + \left[2\zeta\left(4\mathcal{F}\vec{E} + 7c\mathcal{G}\vec{B}\right)\right] \\ &\equiv \epsilon_0 \vec{E} + \vec{P} \end{aligned} \tag{2.6}$$

In eqn. (2.6), we define $\vec{P} \equiv 2\zeta\left(4\mathcal{F}\vec{E} + 7c\mathcal{G}\vec{B}\right)$ as the electric polarization vector.

$$\begin{aligned}\vec{H} = -\frac{\partial \mathcal{L}}{\partial \vec{B}} &= c^2\left(\epsilon_0 + 8\zeta\mathcal{F}\right)\vec{B} - 14\zeta\mathcal{G}c\vec{E} \\ &= \epsilon_0 c^2 \vec{B} + \zeta\left(8c^2\mathcal{F}\vec{B} - 14c\mathcal{G}\vec{E}\right) \\ &\equiv \frac{\vec{B}}{\mu_0} - \vec{M}\end{aligned} \tag{2.7}$$

In eqn. (2.7), we define $\vec{M} \equiv \zeta\left(-8c^2\mathcal{F}\vec{B} + 14c\mathcal{G}\vec{E}\right)$ as the magnetic polarization vector.

These two results are remarkably interesting. They suggest that the vacuum itself behaves like a polarizable medium. Although the vacuum contains no real particles; quantum mechanics says that virtual particles do exist for a very short time. This shows that these virtual particles do interact but this interaction is very small, of the order ζ.

2.1.2 Expressions for $\vec{E}$ and $\vec{B}$

By rewriting eqns. (2.4) and (2.5) in matrix form and inverting this matrix equation, we can obtain an expressions for $\vec{E}$ and $\vec{B}$ in terms of the auxiliary fields $\vec{D}$ and $\vec{H}$.

$$\begin{bmatrix}\vec{D}\\ \vec{H}\end{bmatrix} = \begin{bmatrix}\frac{\partial \mathcal{L}}{\partial \mathcal{F}} & c\frac{\partial \mathcal{L}}{\partial \mathcal{G}} \\ -c\frac{\partial \mathcal{L}}{\partial \mathcal{G}} & c^2\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\end{bmatrix}\begin{bmatrix}\vec{E}\\ \vec{B}\end{bmatrix}$$

$$\begin{bmatrix}\vec{E}\\ \vec{B}\end{bmatrix} = \frac{1}{c^2\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)^2 + c^2\left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)^2}\begin{bmatrix}c^2\frac{\partial \mathcal{L}}{\partial \mathcal{F}} & -c\frac{\partial \mathcal{L}}{\partial \mathcal{G}} \\ c\frac{\partial \mathcal{L}}{\partial \mathcal{G}} & \frac{\partial \mathcal{L}}{\partial \mathcal{F}}\end{bmatrix}\begin{bmatrix}\vec{D}\\ \vec{H}\end{bmatrix}$$

Thus,

$$\vec{E} = \frac{\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)\vec{D} - \frac{1}{c}\left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)\vec{H}}{\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)^2 + \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)^2}$$

$$\vec{B} = \frac{1}{c}\frac{\left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)\vec{D} + \frac{1}{c}\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)\vec{H}}{\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)^2 + \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)^2}$$

Substituting the explicit values for $\frac{\partial \mathcal{L}}{\partial \mathcal{F}} = (\epsilon_0 + 8\zeta\mathcal{F})$ and $\frac{\partial \mathcal{L}}{\partial \mathcal{G}} = 14\zeta\mathcal{G}$, we obtain obtain expressions for $\vec{E}$ and $\vec{B}$. It is also noted, from the definition of ζ (eqn. (1.4)), that ζ is very small, of the order 10^{-52} in SI units. It is thus sufficient for us to only consider terms of the order ζ, since terms with higher orders in ζ would be even smaller and hardly noticeable.

$$\begin{aligned}
\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)^2 &= (\epsilon_0 + 8\zeta\mathcal{F})^2 \\
&= \epsilon_0^2 + 16\epsilon_0\zeta\mathcal{F} + 64\zeta^2\mathcal{F}^2 \\
&\approx \epsilon_0^2 + 16\epsilon_0\zeta\mathcal{F}
\end{aligned}$$

$$\begin{aligned}
\left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)^2 &= 196\zeta^2\mathcal{G}^2 \\
&\approx 0
\end{aligned}$$

$$\begin{aligned}
\frac{1}{\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)^2 + \left(\frac{\partial \mathcal{L}}{\partial \mathcal{G}}\right)^2} &\approx \frac{1}{\left(\frac{\partial \mathcal{L}}{\partial \mathcal{F}}\right)^2} \\
&\approx \frac{1}{\epsilon_0^2\left(1 + 16\frac{\zeta}{\epsilon_0}\mathcal{F}\right)} \\
&\approx \frac{1 - 16\frac{\zeta}{\epsilon_0}\mathcal{F}}{\epsilon_0^2} \qquad (2.8)
\end{aligned}$$

Eqn. (2.8) uses the well known Taylor expansion of $\frac{1}{1+x}$ with the condition $x \ll 1$:

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 - \frac{1}{2}x^3 \cdots \\
&\approx 1 - x
\end{aligned}$$

The resulting expression for $\vec{E}$ and $\vec{B}$ are thus given by:

$$\begin{aligned}
\vec{E} &= \frac{1}{\epsilon_0^2}\left[1 - 16\frac{\zeta}{\epsilon_0}\mathcal{F}\right]\left\{(\epsilon_0 + 8\zeta\mathcal{F})\,\vec{D} - \frac{1}{c}\,(14\zeta\mathcal{G})\,\vec{H}\right\} \\
&\approx \frac{1}{\epsilon_0^2}\left\{(\epsilon_0 + 8\zeta\mathcal{F} - 16\zeta\mathcal{F})\,\vec{D} - \frac{1}{c}14\zeta\mathcal{G}\vec{H}\right\} \\
&= \left(\frac{1}{\epsilon_0} - 8\frac{\zeta}{\epsilon_0^2}\mathcal{F}\right)\vec{D} - \frac{1}{c\epsilon_0^2}\,(14\zeta\mathcal{G})\,\vec{H} \qquad (2.9)
\end{aligned}$$

$$\begin{aligned}
\vec{B} &= \frac{1}{c\epsilon_0^2}\left[1 - 16\frac{\zeta}{\epsilon_0}\mathcal{F}\right]\left\{(14\zeta\mathcal{G})\,\vec{D} + \frac{1}{c}\,(\epsilon_0 + 8\zeta\mathcal{F})\,\vec{H}\right\} \\
&\approx \frac{1}{c\epsilon_0^2}\left\{(14\zeta\mathcal{G})\,\vec{D} + \frac{\epsilon_0}{c}\vec{H} - 16\frac{\zeta}{c}\mathcal{F}\vec{H} + 8\frac{\zeta}{c}\mathcal{F}\vec{H}\right\} \\
&= \frac{1}{c\epsilon_0^2}\,(14\zeta\mathcal{G})\,\vec{D} + \frac{1}{c^2}\left(\frac{1}{\epsilon_0} - 8\frac{\zeta}{\epsilon_0^2}\mathcal{F}\right)\vec{H} \qquad (2.10)
\end{aligned}$$

2.1.3 General Solutions using Maxwell's Equations

Maxwell's equations for quasi-static background electric and magnetic fields are given by:

$$\vec{\nabla}\cdot\vec{D} = j_0 \qquad \vec{\nabla}\times\vec{E} = 0 \tag{2.11}$$

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{H} = \vec{j} \tag{2.12}$$

where j_0 is the charge and $\vec{j}$ is the current. It must also be noted that these are the time-independent equations. The general solutions for $\vec{D}$ and $\vec{H}$ can be written as :

$$\vec{D} = \vec{D}_M + \vec{\nabla}\times\vec{K} \tag{2.13}$$

$$\vec{H} = \vec{H}_M + \vec{\nabla}\phi \tag{2.14}$$

In the above equations (2.13 and 2.14), $\vec{D}_M$ and $\vec{H}_M$ refer to the general solutions of the Maxwell theory, namely the background electric and magnetic fields as well as those fields produced by the charge, j_0, and the current, $\vec{j}$. All other terms are contained in $\vec{\nabla}\times\vec{K}$ and $\vec{\nabla}\phi$. The form of these terms ensure that $\vec{\nabla}\cdot\vec{D} = j_0$, $\vec{\nabla}\times\vec{H} = \vec{j}$, $\vec{\nabla}\times\vec{D} = 0$ and $\vec{\nabla}\cdot\vec{H} = 0$. Considering the quasi-static situation, we introduce a constant electric background field ($\vec{E}_0$) as well as a constant magnetic background field ($\vec{B}_0$) to produce expressions for $\vec{D}_M$ and $\vec{H}_M$:

$$\vec{D}_M = \epsilon_0\vec{E}_0 - \frac{1}{4\pi}\vec{\nabla}_x\int\frac{j_0(\vec{y})}{|\vec{x}-\vec{y}|}\mathrm{d}^3y$$

$$\vec{H}_M = \frac{\vec{B}_0}{\mu_0} + \frac{1}{4\pi}\vec{\nabla}_x\times\int\frac{\vec{j}(\vec{y})}{|\vec{x}-\vec{y}|}\mathrm{d}^3y$$

Consider $\vec{\nabla}\cdot\vec{B} = 0$ (eqn. (2.12)),

$$\vec{\nabla}\cdot\vec{B} = 0 = \frac{14\zeta}{\epsilon_0^2 c}\vec{\nabla}\cdot\left(\mathcal{G}\vec{D}\right) + \frac{1}{c^2}\left\{\frac{1}{\epsilon_0}\vec{\nabla}\cdot\vec{H} - \frac{8\zeta}{\epsilon_0^2}\vec{\nabla}\cdot\left(\mathcal{F}\vec{H}\right)\right\}$$

Substituting $\vec{\nabla}\cdot\vec{H} = \nabla^2\phi$ gives:

$$0 = \vec{\nabla}\cdot\left\{\frac{14\zeta}{\epsilon_0^2 c}\mathcal{G}\vec{D} - \frac{8\zeta}{\epsilon_0^2 c^2}\mathcal{F}\vec{H}\right\} + \frac{1}{\epsilon_0 c^2}\nabla^2\phi$$

$$\vec{\nabla}\cdot\left\{\frac{14\zeta}{\epsilon_0^2 c}\mathcal{G}\vec{D} - \frac{8\zeta}{\epsilon_0^2 c^2}\mathcal{F}\vec{H}\right\} = -\vec{\nabla}\cdot\left\{\frac{1}{\epsilon_0 c^2}\vec{\nabla}\phi\right\} \tag{2.15}$$

Following Helmholtz's Theorem [9], the solution for $\frac{1}{\epsilon_0 c^2}\vec{\nabla}\phi$ in eqn. (2.15) can be expressed as:

$$\frac{1}{\epsilon_0 c^2}\vec{\nabla}\phi(\vec{x}) = \frac{1}{4\pi}\vec{\nabla}_x\int\mathrm{d}^3y\frac{\vec{\nabla}_y\cdot\left\{\frac{14\zeta}{\epsilon_0^2 c}\mathcal{G}\vec{D} - \frac{8\zeta}{\epsilon_0^2 c^2}\mathcal{F}\vec{H}\right\}}{|\vec{x}-\vec{y}|} \tag{2.16}$$

A similar treatment can be used on $\vec{\nabla}\times\vec{E} = 0$ (eqn. (2.11)) such that:

$$\vec{\nabla}\times\left(\frac{1}{\epsilon_0}\vec{\nabla}\times\vec{K}\right) = \vec{\nabla}\times\left\{\frac{8\zeta}{\epsilon_0^2}\mathcal{F}\vec{D} + \frac{14\zeta}{\epsilon_0^2 c}\mathcal{G}\vec{H}\right\} \tag{2.17}$$

$$\vec{\nabla} \times \vec{K}(\vec{x}) = \frac{\epsilon_0}{4\pi} \vec{\nabla}_x \times \int \mathrm{d}^3 y \frac{\vec{\nabla}_y \times \left\{ \frac{8\zeta}{\epsilon_0^2} \mathcal{F} \vec{D} + \frac{14\zeta}{\epsilon_0^2 c} \mathcal{G} \vec{H} \right\}}{|\vec{x} - \vec{y}|} \tag{2.18}$$

It is also noted from eqns. (2.16) and (2.18) that $\vec{\nabla}\phi(\vec{x})$ and $\vec{\nabla} \times \vec{K}(\vec{x})$ are both at least of the order ζ.

2.1.4 Obtaining the Induced Fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$

We now define the induced electric field ($\vec{\mathcal{E}}(\vec{x})$) as the difference between the electric field (as defined in eqn. (2.9)) and the Maxwell electric field.

$$\vec{\mathcal{E}}(\vec{x}) \equiv \vec{E}(\vec{x}) - \frac{1}{\epsilon_0} \vec{D}_M(\vec{x}) \tag{2.19}$$

Similarly, the induced magnetic field ($\vec{\mathcal{B}}(\vec{x})$) can be defined as the difference between the magnetic field (as defined in eqn. (2.10)) and the Maxwell magnetic field.

$$\vec{\mathcal{B}}(\vec{x}) \equiv \vec{B}(\vec{x}) - \mu_0 \vec{H}_M(\vec{x}) \tag{2.20}$$

Using eqns. (2.9) and (2.13), we can express $\vec{E}$ as:

$$\vec{E} = \frac{1}{\epsilon_0} \vec{D}_M + \frac{1}{\epsilon_0} \vec{\nabla} \times \vec{K} + \left\{ -\frac{8\zeta}{\epsilon_0^2} \mathcal{F} \vec{D} - \frac{14\zeta}{\epsilon_0^2 c} \mathcal{G} \vec{H} \right\} \tag{2.21}$$

Thus,

$$\vec{\mathcal{E}}(\vec{x}) = \frac{1}{\epsilon_0} \vec{\nabla} \times \vec{K} - \zeta \left\{ \frac{8}{\epsilon_0^2} \mathcal{F} \vec{D} + \frac{14}{\epsilon_0^2 c} \mathcal{G} \vec{H} \right\} \tag{2.22}$$

$$\begin{aligned} \vec{\nabla} \times \vec{\mathcal{E}}(\vec{x}) &= \frac{1}{\epsilon_0} \vec{\nabla} \times \vec{\nabla} \times \vec{K} - \zeta \vec{\nabla} \times \left\{ \frac{8}{\epsilon_0^2} \mathcal{F} \vec{D} + \frac{14}{\epsilon_0^2 c} \mathcal{G} \vec{H} \right\} \\ &= 0 \qquad \text{, using eqn (2.17)} \end{aligned} \tag{2.23}$$

This result is not surprising since $\vec{\nabla} \times \vec{E} = 0$ and $\vec{\nabla} \times \vec{D}_M = 0$.

$$\vec{\nabla} \cdot \vec{\mathcal{E}}(\vec{x}) = -\zeta \vec{\nabla} \cdot \left\{ \frac{8}{\epsilon_0^2} \mathcal{F} \vec{D} + \frac{14}{\epsilon_0^2 c} \mathcal{G} \vec{H} \right\} \tag{2.24}$$

Using Helmholtz's theorem [9] and the fact that $\vec{\nabla} \times \vec{\mathcal{E}} = 0$, it can be shown that

$$\vec{\mathcal{E}} = -\vec{\nabla}\psi \tag{2.25}$$

where, ψ is a scalar function. Helmholtz's theorem also gives a general solution for ψ:

$$\psi = \frac{1}{4\pi} \int \mathrm{d}^3 y \frac{\vec{\nabla} \cdot \vec{\mathcal{E}}}{|\vec{x} - \vec{y}|} \tag{2.26}$$

Bringing eqns. (2.24), (2.25) and (2.26) together gives us an expression for $\vec{\mathcal{E}}$:

$$\vec{\mathcal{E}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2} \vec{\nabla}_x \int \frac{\mathrm{d}^3 y}{|\vec{x} - \vec{y}|} \vec{\nabla}_y \cdot \left\{ 8\mathcal{F}\vec{D} + \frac{14}{c} \mathcal{G} \vec{H} \right\} \tag{2.27}$$

Consider $\zeta\mathcal{F}\vec{D}$ and $\zeta\mathcal{G}\vec{H}$; only keeping terms of the order ζ ($\mathcal{O}(\zeta)$)

$$\begin{aligned}
\zeta\mathcal{F}\vec{D} &= \zeta\left(E^2 - c^2B^2\right)\vec{D} \\
&= \zeta\left(\left(\vec{E}_M + \vec{\mathcal{E}}\right)^2 - c^2\left(\vec{B}_M + \vec{\mathcal{B}}\right)^2\right)\vec{D} && \text{,eqns. (2.19) and (2.20)} \\
&= \zeta\left(E_M^2 + 2\vec{\mathcal{E}}\cdot\vec{E}_M + \mathcal{E}^2 - c^2B_M^2 - 2c^2\vec{B}_M\cdot\vec{\mathcal{B}} - c^2\mathcal{B}^2\right)\vec{D} \\
&= \zeta\left(E_M^2 - c^2B_M^2\right)\vec{D} + \mathcal{O}(\zeta^2) && \text{,}\vec{\mathcal{E}}\text{ and }\vec{\mathcal{B}}\text{ are }\mathcal{O}(\zeta) \\
&\approx \zeta\mathcal{F}_M\vec{D} \\
&= \zeta\mathcal{F}_M\left(\vec{D}_M + \vec{\nabla}\times\vec{K}\right) && \text{,eqn. (2.13)} \\
&= \zeta\mathcal{F}_M\vec{D}_M + \mathcal{O}(\zeta^2) && \text{, }\vec{\nabla}\times\vec{K}\text{ is }\mathcal{O}(\zeta)\text{ (eqn. (2.18))} \\
&\approx \zeta\mathcal{F}_M\vec{D}_M && (2.28)
\end{aligned}$$

$$\begin{aligned}
\zeta\mathcal{G}\vec{H} &= c\zeta\left(\vec{E}\cdot\vec{B}\right)\vec{H} \\
&= c\zeta\left(\vec{E}_M + \vec{\mathcal{E}}\right)\cdot\left(\vec{B}_M + \vec{\mathcal{B}}\right)\vec{H} && \text{,eqns. (2.19) and (2.20)} \\
&= c\zeta\left[\vec{E}_M\cdot\vec{B}_M + \vec{E}_M\cdot\vec{\mathcal{B}} + \vec{\mathcal{E}}\cdot\vec{B}_M + \vec{\mathcal{E}}\cdot\vec{\mathcal{B}}\right]\vec{H} \\
&= c\zeta\vec{E}_M\cdot\vec{B}_M + \mathcal{O}(\zeta^2) && \text{,}\vec{\mathcal{E}}\text{ and }\vec{\mathcal{B}}\text{ are }\mathcal{O}(\zeta) \\
&\approx \zeta\mathcal{G}_M\vec{H} \\
&= \zeta\mathcal{G}_M\left(\vec{H}_M + \vec{\nabla}\phi\right) && \text{,eqn. (2.13)} \\
&= \zeta\mathcal{G}_M\vec{H}_M + \mathcal{O}(\zeta^2) && \text{,}\vec{\nabla}\phi\text{ is }\mathcal{O}(\zeta)\text{ (eqn. (2.16))} \\
&\approx \zeta\mathcal{G}_M\vec{H}_M && (2.29)
\end{aligned}$$

In eqns. (2.28) and (2.29), $\mathcal{F}_M$ and $\mathcal{G}_M$ refer to the Lorentz invariants $\mathcal{F}$ and $\mathcal{G}$ when the Maxwell fields are the only fields present. The final form of the induced electric field is thus:

$$\vec{\mathcal{E}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x\int\frac{\mathrm{d}^3y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y\cdot\left\{8\mathcal{F}_M\vec{D}_M + \frac{14}{c}\mathcal{G}_M\vec{H}_M\right\} \tag{2.30}$$

A similar procedure is used to obtain an expression for the induced magnetic field. Using eqns. (2.10) and (2.14), $\vec{B}$ can be expressed as:

$$\vec{B} = \mu_0\vec{H}_M + \mu_0\vec{\nabla}\phi + \left\{-\frac{8\zeta}{c^2\epsilon_0^2}\mathcal{F}\vec{H} + \frac{14\zeta}{\epsilon_0 c}\mathcal{G}\vec{D}\right\} \tag{2.31}$$

Thus,

$$\vec{\mathcal{B}}(\vec{x}) = \mu_0\vec{\nabla}\phi + \left\{-\frac{8\zeta}{\epsilon_0^2c^2}\mathcal{F}\vec{H} + \frac{14\zeta}{\epsilon_0^2 c}\mathcal{G}\vec{D}\right\} \tag{2.32}$$

Using a similar procedure as that used to obtain $\vec{\nabla}\times\vec{\mathcal{E}} = 0$, it can be show that $\vec{\nabla}\cdot\vec{\mathcal{B}} = 0$ using eqn. (2.15). Calculating $\vec{\nabla}\cdot\vec{\mathcal{B}}$ gives :

$$\vec{\nabla}\times\vec{\mathcal{B}}(\vec{x}) = \frac{\zeta}{\epsilon_0^2 c}\vec{\nabla}\times\left\{-\frac{8}{c}\mathcal{F}\vec{H} + 14\mathcal{G}\vec{D}\right\} \tag{2.33}$$

Using Helmholtz's Theorem and the fact that $\vec{\nabla} \cdot \vec{\mathcal{B}} = 0$, $\mathcal{B}$ can be expressed as:

$$\vec{\mathcal{B}} = \vec{\nabla} \times \vec{A} \tag{2.34}$$

where $\vec{A}$ is a vector function and is given by :

$$\vec{A} = \frac{1}{4\pi} \int \mathrm{d}^3 y \frac{\vec{\nabla} \times \vec{\mathcal{B}}}{|\vec{x} - \vec{y}|} \tag{2.35}$$

Bringing eqns. (2.33), (2.34) and (2.35) together produces the expression :

$$\vec{\mathcal{B}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2 c} \vec{\nabla}_x \times \int \frac{\mathrm{d}^3 y}{|\vec{x} - \vec{y}|} \vec{\nabla}_y \times \left\{ -\frac{8}{c} \mathcal{F} \vec{H} + 14 \mathcal{G} \vec{D} \right\} \tag{2.36}$$

Similarly to eqns. (2.28) and (2.29), $\mathcal{F}\vec{H}$ and $\mathcal{G}\vec{D}$ can be replaced by $\mathcal{F}_M \vec{H}_M$ and $\mathcal{G}_M \vec{D}_M$ respectively without affecting $\vec{\mathcal{B}}$ to leading order in ζ. The final form of the induced magnetic field is thus:

$$\vec{\mathcal{B}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2 c} \vec{\nabla}_x \times \int \frac{\mathrm{d}^3 y}{|\vec{x} - \vec{y}|} \vec{\nabla}_y \times \left\{ -\frac{8}{c} \mathcal{F}_M \vec{H}_M + 14 \mathcal{G}_M \vec{D}_M \right\} \tag{2.37}$$

2.2 Summary and Discussion

In deriving the equations for the induced electric and magnetic field, we find that the electric displacement (eqn. (2.6)) and the magnetization (eqn. (2.7)) can be expressed as follows:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$$
$$\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$$

where,

$$\vec{P} = 2\zeta \left(4\mathcal{F}\vec{E} + 7c\mathcal{G}\vec{B} \right)$$
$$\vec{M} = 2\zeta \left(4c^2 \mathcal{F}\vec{B} - 7c\mathcal{G}\vec{E} \right)$$

The derivation of $\vec{D}$ and $\vec{H}$ did not assume the presence of a medium but here it can be seen that there are terms that look like the electric polarization vector ($\vec{P}$) and magnetic polarization vector ($\vec{M}$). This strange behaviour can be understood as the vacuum behaving as a polarized medium. This polarization is very small since $\vec{P}$ and $\vec{M}$ are both proportional to ζ. As $\zeta \to 0$, $\vec{P}$ and $\vec{M}$ will vanish. The main result is the derivation of the equations for the induced electric field (eqn. (2.30)) and the induced magnetic field (eqn. (2.37)):

$$\vec{\mathcal{E}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2} \vec{\nabla}_x \int \frac{\mathrm{d}^3 y}{|\vec{x} - \vec{y}|} \vec{\nabla}_y \cdot \left\{ 8\mathcal{F}_M \vec{D}_M + \frac{14}{c} \mathcal{G}_M \vec{H}_M \right\}$$
$$\vec{\mathcal{B}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2 c} \vec{\nabla}_x \times \int \frac{\mathrm{d}^3 y}{|\vec{x} - \vec{y}|} \vec{\nabla}_y \times \left\{ -\frac{8}{c} \mathcal{F}_M \vec{H}_M + 14 \mathcal{G}_M \vec{D}_M \right\}$$

It must be noted that the quantum effects are contained within the factor ζ. On the other hand, the factors $\mathcal{F}_M$, $\mathcal{G}_M$, $\vec{D}_M$ and $\vec{H}_M$ are all obtained from classical sources and external fields. This is very handy, since $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ can now be handled as classical time-independent objects.

Chapter 3

Investigating the Induced Fields

3.1 Charged Shell in a Quasi-static Magnetic Field

3.1.1 The Situation

Consider a charged spherical shell of radius R, which carries a uniform surface charge, in the presence of an external magnetic field, $\vec{B}_0$ (see Figure 3.1). The electric field produced by the spherical shell is known and is given by:

$$\vec{E}_M(\vec{x}) = \frac{1}{\epsilon_0}\vec{D}_M = \frac{Q}{4\pi\epsilon_0}\frac{\Theta(r-R)}{r^2}\hat{e}_r \tag{3.1}$$

where,

$\vec{x} = (r, \theta, \phi)$	- position vector in spherical coordinates
r	- radius with the unit vector $\hat{e}_r$
θ	- polar angle with the unit vector $\hat{e}_\theta$
ϕ	- azimuthal angle with the unit vector $\hat{e}_\phi$
Q	- total charge on the shell

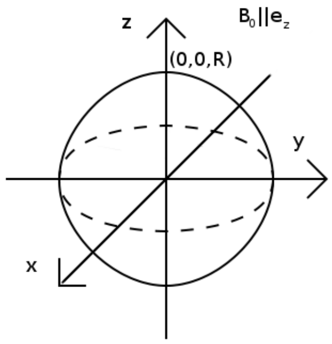

Figure 3.1: A charged shell in an external magnetic field

The external magnetic field is orientated along the z-axis and has a field strength B_0. $\vec{B}_0$ is given by:

$$\vec{B}_0 = B_0\hat{e}_z = B_0\left(\cos(\theta)\hat{e}_r - \sin(\theta)\hat{e}_\theta\right) \tag{3.2}$$

Eqns. (3.1) and (3.2) give rise to the following expressions:

$$\begin{aligned}
\mathcal{F}_M &= \frac{1}{2}\left[E_M^2 + c^2B_0^2\right] \\
&= \frac{1}{2}\left[\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{1}{r^4}\Theta(r-R) - c^2B_0^2\right] && (3.3)\\
\mathcal{G}_M &= c\vec{E}_M\cdot\vec{B}_0 \\
&= \frac{cQB_0}{4\pi\epsilon_0}\frac{\cos(\theta)}{r^2}\Theta(r-R) && (3.4)\\
\vec{D}_M &= \frac{Q}{4\pi}\frac{1}{r^2}\Theta(r-R)\hat{e}_r && (3.5)\\
\vec{H}_M &= \frac{B_0}{\mu_0}\left(\cos(\theta)\hat{e}_r - \sin(\theta)\hat{e}_\theta\right) && (3.6)
\end{aligned}$$

Define $\vec{V}_1$ and $\vec{V}_2$ as:

$$\vec{V}_1(\vec{x}) \equiv 4\mathcal{F}_M(\vec{x})\vec{D}_M(\vec{x}) + \frac{7}{c}\mathcal{G}_M(\vec{x})\vec{H}_M(\vec{x}) \tag{3.7}$$

$$\vec{V}_2(\vec{x}) \equiv -\frac{4}{c}\mathcal{F}_M(\vec{x})\vec{H}_M(\vec{x}) + 7\mathcal{G}_M(\vec{x})\vec{D}_M(\vec{x}) \tag{3.8}$$

such that eqns. (2.30) and (2.37) become:

$$\vec{\mathcal{E}}(\vec{x}) = \frac{2\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x\int\frac{\mathrm{d}^3y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y\cdot\vec{V}_1(\vec{y}) \tag{3.9}$$

$$\vec{\mathcal{B}}(\vec{x}) = \frac{2\zeta}{4\pi\epsilon_0^2c}\vec{\nabla}_x\times\int\frac{\mathrm{d}^3y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y\times\vec{V}_2(\vec{y}) \tag{3.10}$$

3.1.2 Calculating $\vec{\mathcal{E}}(\vec{x})$

Substituting eqns. (3.3), (3.4), (3.5) and (3.6) into eqn. (3.7):

$$\begin{aligned}
\vec{V}_1(\vec{x}) =& 2\frac{Q}{4\pi}\left[\left(\frac{Q}{4\pi\epsilon_0^2}\right)^2\frac{1}{r^6}\Theta(r-R)\hat{e}_r - c^2B_0^2\frac{1}{r^2}\Theta(r-R)\hat{e}_r\right]\\
&+ 7\frac{QB_0^2}{4\pi\epsilon_0\mu_0}\left[\frac{\cos^2(\theta)}{r^2}\Theta(r-R)\hat{e}_r - \frac{\cos(\theta)\sin(\theta)}{r^2}\Theta(r-R)\hat{e}_\theta\right]
\end{aligned}$$

Thus,

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{V}_1(\vec{x}) = \; & 2\frac{Q}{4\pi}\left[\left(\frac{Q}{4\pi\epsilon_0}\right)^2\vec{\nabla}\cdot\left(\frac{1}{r^6}\Theta(r-R)\hat{e}_r\right) - c^2B_0^2\vec{\nabla}\cdot\left(\frac{1}{r^2}\Theta(r-R)\hat{e}_r\right)\right] \\
& + 7\frac{QB_0^2}{4\pi\epsilon_0\mu_0}\left[\vec{\nabla}\cdot\left(\frac{\cos^2(\theta)}{r^2}\Theta(r-R)\hat{e}_r\right) - \vec{\nabla}\cdot\left(\frac{\cos(\theta)\sin(\theta)}{r^2}\Theta(r-R)\hat{e}_\theta\right)\right] \\
= \; & 2\frac{Q}{4\pi}\left[\left(\frac{Q}{4\pi\epsilon}\right)^2\left\{\frac{\delta(r-R)}{r^6} - 4\frac{\Theta(r-R)}{r^7}\right\} - c^2B_0^2\frac{\delta(r-R)}{r^2}\right] \\
& + 7\frac{QB_0^2}{4\pi\epsilon_0\mu_0}\left[\frac{\cos^2(\theta)}{r^2}\delta(r-R) - \frac{3\cos^2(\theta)-1}{r^3}\Theta(r-R)\right]
\end{aligned} \tag{3.11}
$$

We now introduce the following notation:

- $\vec{y} = (r', \theta', \phi')$ as a position vector
- $\Omega_x = (\theta, \phi)$
- $\Omega_y = (\theta', \phi')$
- $Y_{lm}(\Omega_x) = Y_{lm}$ is the spherical harmonic of degree l and order m
- $Y_{lm}(\Omega_y) = Y'_{lm}$

Using a table of spherical harmonics [10], we can introduce spherical harmonic functions into eqn. (3.11):

$$
\begin{aligned}
1 &= Y_{00}\sqrt{4\pi} \\
\cos^2(\theta) &= \frac{1}{3}\sqrt{\frac{16\pi}{5}}Y_{20} + \frac{1}{3}\sqrt{4\pi}Y_{00} \\
3\cos^2(\theta) - 1 &= \sqrt{\frac{16\pi}{5}}Y_{20}
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{V}_1(\vec{x}) = \; & 2\frac{Q}{4\pi}\left[\left(\frac{Q}{4\pi\epsilon}\right)^2\left\{\frac{\delta(r-R)}{r^6} - 4\frac{\Theta(r-R)}{r^7}\right\} - c^2B_0^2\frac{\delta(r-R)}{r^2}\right]Y_{00}\sqrt{4\pi} \\
& + 7\frac{QB_0^2}{4\pi\epsilon_0\mu_0}\left[\frac{1}{3}\frac{\delta(r-R)}{r^2}\left\{\sqrt{\frac{16\pi}{5}}Y_{20} + \sqrt{4\pi}Y_{00}\right\} - \frac{\Theta(r-R)}{r^3}\sqrt{\frac{16\pi}{5}}Y_{20}\right]
\end{aligned} \tag{3.12}
$$

Using the following identity, we can rewrite eqn. (3.9):

$$
\frac{1}{|\vec{x}-\vec{y}|} = \sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}(\Omega_x)Y_{lm}^*(\Omega_y) \tag{3.13}
$$

Note that if $r > r'$ then $r_> = r$, conversely if $r < r'$ then $r_> = r'$.
$\vec{\mathcal{E}}(\vec{x})$ is thus,

$$
\vec{\mathcal{E}}(\vec{x}) = \frac{2\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x\sum_{l,m}\frac{4\pi}{2l+1}\int_0^\infty \mathrm{d}r'\, r'^2\frac{r_<^l}{r_>^{l+1}}\int \mathrm{d}\Omega_y Y_{lm}Y_{lm}^{'*}\vec{\nabla}\cdot\vec{V}_1(\vec{y}) \tag{3.14}
$$

The orthogonality relation of spherical harmonic functions is given by:

$$\int \mathrm{d}\Omega\, Y_{lm}(\Omega) Y_{l'm'}(\Omega) = \delta_{ll'}\delta_{mm'} \tag{3.15}$$

Substituting eqn. (3.12) into eqn. (3.14) and applying eqn. (3.15) gives:

$$\begin{aligned}
\vec{\mathcal{E}}(\vec{x}) =& \frac{2\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x \sum_{l,m} \frac{4\pi}{2l+1} Y_{lm} \int_0^\infty \mathrm{d}r'\, r'^2 \frac{r_<^l}{r_>^{l+1}} \left\{ 2\frac{Q}{4\pi}\left[\left(\frac{Q}{4\pi\epsilon}\right)^2 \left\{\frac{\delta(r'-R)}{r'^6} - 4\frac{\Theta(r'-R)}{r'^7}\right\}\right.\right. \\
& \left. -c^2 B_0^2 \frac{\delta(r'-R)}{r'^2}\right] \delta_{l0}\delta_{m0}\sqrt{4\pi} + 7\frac{QB_0^2}{4\pi\epsilon_0\mu_0}\left[\frac{1}{3}\frac{\delta(r'-R)}{r'^2}\left\{\sqrt{\frac{16\pi}{5}}\delta_{l2}\delta_{m0} + \sqrt{4\pi}\delta_{l0}\delta_{m0}\right\}\right. \\
& \left.\left. -\frac{\Theta(r'-R)}{r'^3}\sqrt{\frac{16\pi}{5}}\delta_{l2}\delta_{m0}\right]\right\} \\
=& \frac{2\zeta}{\epsilon_0^2}\vec{\nabla}_x \left\{ 2\frac{Q}{4\pi}\left[\left(\frac{Q}{4\pi\epsilon}\right)^2 \left\{\int_0^\infty \mathrm{d}r' \frac{\delta(r'-R)}{rr'^4} - 4\int_0^\infty \mathrm{d}r' \frac{\Theta(r'-R)}{r_> r'^5}\right\}\right.\right. \\
& \left. -c^2 B_0^2 \int_0^\infty \mathrm{d}r' \frac{\delta(r'-R)}{r}\right] Y_{00}\sqrt{4\pi} + 7\frac{QB_0^2}{4\pi\epsilon_0\mu_0}\left[\frac{1}{3}\left\{\frac{1}{5}\sqrt{\frac{16\pi}{5}}\int_0^\infty \mathrm{d}r' \frac{r'^2\delta(r'-R)}{r^3} Y_{20}\right.\right. \\
& \left.\left.\left. +\sqrt{4\pi}\int_0^\infty \mathrm{d}r' \frac{\delta(r'-R)}{r} Y_{00}\right\} - \int_0^\infty \mathrm{d}r' \frac{r_<\Theta(r'-R)}{r_>^2 r'}\sqrt{\frac{16\pi}{5}} Y_{20}\right]\right\}
\end{aligned}$$

Note that $\vec{\mathcal{E}}(\vec{x})$ will always be evaluated at a point far away from the surface of the shell. Therefore terms with $\delta(r'-R)$ as factors will always have $r_> = r$. After integrating and simplifying $\vec{\mathcal{E}}(\vec{x})$ is given by:

$$\vec{\mathcal{E}}(\vec{x}) = -\vec{\nabla}_x \left[-\frac{4\zeta}{5\epsilon_0^4}\left(\frac{Q}{4\pi}\right)^3 \frac{1}{r^5} - \frac{QB_0^2\zeta}{4\pi\epsilon_0^3\mu_0}\left\{\frac{7}{3}\left(3\cos^2(\theta)-1\right)\frac{R^2}{r^3} - \left(7\cos^2(\theta)-3\right)\frac{1}{r}\right\}\right] \tag{3.16}$$

Defining the scalar functions

$$\begin{aligned}
V_1 &= -\frac{4\zeta}{5\epsilon_0^4}\left(\frac{Q}{4\pi}\right)^3 \frac{1}{r^5} \\
V_2 &= -\frac{7QB_0^2\zeta}{12\pi\epsilon_0^3\mu_0}\left(3\cos^2(\theta)-1\right)\frac{R^2}{r^3} \\
V_3 &= \frac{QB_0^2\zeta}{4\pi\epsilon_0^3\mu_0}\left(7\cos^2(\theta)-3\right)\frac{1}{r}
\end{aligned}$$

Such that $\vec{\mathcal{E}}(\vec{x}) = -\vec{\nabla}_x\left[V_1+V_2+V_3\right]$.
The scalar potential V of a charged spherical shell of radius R and charge Q is given by:

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q}{r}$$

Using $\zeta \approx 10^{-52}$, $\epsilon_0 \approx 10^{-11}$ and $\mu_0 \approx 10^{-6}$ (in SI units); we can now compare the relative strengths of the scalar functions V_1,V_2 and V_3:

$$\begin{aligned}
\frac{V_1}{V} &= -\frac{4\zeta}{5\epsilon_0^2(4\pi)^2}\frac{Q^2}{r^4} \approx -10^{-20}\frac{Q^2}{r^4} \\
\frac{V_2}{V} &= -\frac{7}{3}(\cos^2(\theta)-1)\frac{\zeta}{\epsilon_0^2\mu_0}\frac{B_0^2R^2}{r^2} \approx -10^{-24}\frac{B_0^2R^2}{r^2} \\
\frac{V_3}{V} &= (7\cos^2(\theta)-3)\frac{\zeta}{\epsilon_0^2\mu_0}B_0^2 \approx 10^{-24}B_0^2
\end{aligned}$$

From these relations, it is clear that the V_1, V_2 and V_3 are very much smaller than the electric potential of the charged spherical shell. This means that $\vec{\mathcal{E}}(\vec{x})$ will be overpowered by the more powerful electric field $\vec{E}_M(\vec{x})$.

3.1.3 Calculating $\vec{\mathcal{B}}(\vec{x})$

An explicit expression for $\vec{V}_2$ can be obtained by using eqn. (3.3), (3.4), (3.5) and (3.6)

$$\begin{aligned}
\vec{V}_2 &= -\frac{4}{c}\mathcal{F}_M\vec{H}_M + 7\mathcal{G}_M\vec{D}_M \\
&= \frac{B_0}{\mu_0}\left[5\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{\cos(\theta)}{r^4}\Theta(r-R)\hat{e}_r + 2\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{\sin(\theta)}{r^4}\Theta(r-R)\hat{e}_\theta - 2c^2B_0^2\left(\cos(\theta)\hat{e}_r - \sin(\theta)\hat{e}_\theta\right)\right]
\end{aligned}$$

The curl of $\vec{V}_2$ can now be calculated:

$$\vec{\nabla}\times\vec{V}_2 = \frac{B_0}{\mu_0 c}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left[2\frac{\delta(r-R)}{r^4} - \frac{\Theta(r-R)}{r^5}\right]\sin(\theta)\hat{e}_\phi$$

Using a table of spherical harmonics, it is possible to show that:

$$\begin{aligned}
\sin(\theta)\hat{e}_\phi &= \sin(\theta)\left(-\sin(\phi)\hat{e}_x + \cos(\phi)\hat{e}_y\right) \\
&= -\sin(\theta)\sin(\phi)\hat{e}_x + \sin(\theta)\cos(\phi)\hat{e}_y \\
&= \sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}+Y_{1-1}}{2i}\hat{e}_x - \frac{Y_{11}-Y_{1-1}}{2}\hat{e}_y\right)
\end{aligned}$$

Thus $\vec{\nabla}\times\vec{V}_2$ becomes

$$\vec{\nabla}\times\vec{V}_2 = \frac{B_0}{\mu_0 c}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left[2\frac{\delta(r-R)}{r^4} - \frac{\Theta(r-R)}{r^5}\right]\sqrt{\frac{8\pi}{3}}\left[\frac{Y_{11}+Y_{1-1}}{2i}\hat{e}_x - \frac{Y_{11}-Y_{1-1}}{2}\hat{e}_y\right]$$

Using the same identity (eqn. (3.13)) as was used to calculate $\vec{\mathcal{E}}$ (eqn. (3.14)), $\vec{\mathcal{B}}$ can expressed as:

$$\vec{\mathcal{B}}(\vec{x}) = \frac{2\zeta}{4\pi\epsilon_0^2 c}\vec{\nabla}_x\int d^3y\sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}Y_{lm}Y_{lm}^{\prime *}\vec{\nabla}_y\times\vec{V}_2$$

Thus,

$$\begin{aligned}\vec{\mathcal{B}}(\vec{x}) =& \frac{2\zeta B_0}{\epsilon_0}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \vec{\nabla}_x \times \sum_{l,m}\frac{1}{2l+1}\int_0^\infty \mathrm{d}r'\, r'^2 \frac{r_<^l}{r_>^{l+1}} Y_{lm}\left[2\frac{\delta(r'-R)}{r'^4}\right.\\ &\left.-\frac{\Theta(r'-R)}{r'^5}\right]\sqrt{\frac{8\pi}{3}}\left[\frac{\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}}{2i}\hat{e}_x - \frac{\delta_{l1}\delta_{m1}-\delta_{l1}\delta_{m-1}}{2}\hat{e}_y\right]\\ =& \frac{2\zeta B_0}{3\epsilon_0}\left(\frac{Q}{4\pi\epsilon_0}\right)^2 \vec{\nabla}_x \times \int_0^\infty \mathrm{d}r' \frac{r_<}{r_>^2}\left[2\frac{\delta(r'-R)}{r'^2}\right.\\ &\left.-\frac{\Theta(r'-R)}{r'^3}\right]\sqrt{\frac{8\pi}{3}}\left(\frac{Y_{11}+Y_{1-1}}{2i}\hat{e}_x - \frac{Y_{11}-Y_{1-1}}{2}\hat{e}_y\right)\end{aligned}$$

After completing the integration and simplifying, the final form of $\vec{\mathcal{B}}$ can be expressed as:

$$\vec{\mathcal{B}}(\vec{x}) = \vec{\nabla}_x \times \left[\frac{2\zeta B_0}{3\epsilon_0}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left\{\frac{1}{r^2R}+\frac{3}{4r^3}\right\}\sin(\theta)\hat{e}_\phi\right] \tag{3.17}$$

From eqn. (3.17), one can associate the vector potential $\vec{A}$ with:

$$\vec{A} = \frac{2\zeta B_0}{3\epsilon_0}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left\{\frac{1}{r^2R}+\frac{3}{4r^3}\right\}\hat{e}_z \times \hat{e}_r \tag{3.18}$$

3.1.4 Summary and Discussion

It must be noted that there is a restriction to r. This restriction arises from the fact that $|\vec{E}_M| < 1.3\times 10^{18} V/m$. In other words:

$$\begin{aligned}|\vec{E}_M| = \frac{Q}{4\pi\epsilon_0}\frac{1}{r^2} &< 1.3\times 10^{18}\\ r^2 &> 1.3\times 10^{-18}\frac{Q}{4\pi\epsilon_0}\end{aligned}$$

Thus,

$$r > 8.3\sqrt{Q}\times 10^{-5}$$

where r is measured in meters and Q in coulombs. However, this means that if we use the shell as a model for the proton then we cannot get too close to the proton since its size is of the order $10^{-15}m$ which is less than the restriction $8.3\sqrt{1.9\times 10^{-19}}\times 10^{-5}m \approx 3.6\times 10^{-14}m$.

The induced electric field produced by the interaction of the charged spherical shell and the quasi-static magnetic field is given in eqn. (3.16). This induced electric field is much smaller than the field produced by the charged spherical shell and will therefore be much more difficult to detect.

The induced magnetic field produced by the interaction of the charged spherical shell and quasi-static magnetic field is given by eqn. (3.17):

$$\vec{\mathcal{B}}(\vec{x}) = \vec{\nabla}_x \times \left[\frac{2\zeta B_0}{3\epsilon_0}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left\{\frac{1}{r^2R}+\frac{3}{4r^3}\right\}\sin(\theta)\hat{e}_\phi\right]$$

The vector field that can be associated with the induced magnetic field is given by eqn. (3.18):

$$\vec{A} = \frac{2\zeta B_0}{3\epsilon_0}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\left\{\frac{1}{r^2R}+\frac{3}{4r^3}\right\}\hat{e}_z \times \hat{e}_r$$

If $r \gg R$, which can be expected in an experimental setup, then the $\frac{1}{r^2R}$ term dominates $\frac{3}{4r^3}$. Thus,

$$\vec{A} = \frac{2\zeta}{3\epsilon_0}\left(\frac{Q}{4\pi\epsilon_0}\right)^2\frac{1}{r^2R}\vec{B}_0 \times \hat{e}_r$$

This vector potential is the same as that produced by the magnetic dipole moment:

$$\vec{m} = \frac{c^2\zeta}{6\pi\epsilon_0^2}\frac{Q^2}{R}\vec{B}_0 \tag{3.19}$$

where $\vec{A} = \frac{\mu_0}{4\pi}\frac{\vec{m}\times\hat{e}_r}{r^2}$

Although this induced magnetic dipole is small, of the order ζ, it did not exist before and should therefore be detectable. When considering an experimental setup; the total charge Q, radius of the shell R and the external quasi-static magnetic field $\vec{B}_0$ should be chosen in such a way to maximize the induced magnetic moment $\vec{m}$.

3.2 Spherical Magnetic Dipole in a Quasi-static Electric Field

3.2.1 The Situation

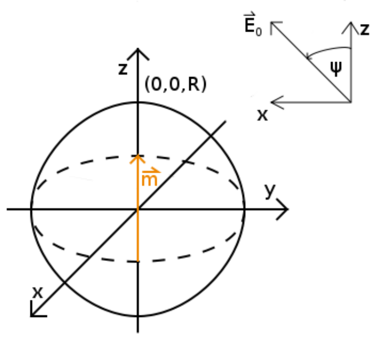

Figure 3.2: A spherical magnetic moment in an external electric field

Consider a spherical shell of radius R with a current $\vec{j} = \frac{3|\vec{m}|}{4\pi R^3}\delta(r-R)\hat{e}_\phi$ moving on the surface of the shell. This current gives rise to the magnetic field

$$\vec{B}_M = \frac{\mu_0}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{e}_r)\hat{e}_r - \vec{m}}{r^3}\Theta(r-R) + \frac{2\vec{m}}{R^3}\Theta(R-r)\right] \tag{3.20}$$

where $\vec{m}$ represents the magnetic dipole moment of the source term $\vec{j}$. $\Theta(r-R)$ is the heaviside step function which is defined by:

$$\Theta(x) = \begin{cases} 1 & : x \geq 0 \\ 0 & : x < 0 \end{cases}$$

At distances $r > R$, the magnetic field $\vec{B}_M$ behaves like a magnetic dipole. To produce $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$, a quasi-static electric field $\vec{E}_0$ is applied over $\vec{B}_M$. By choosing the magnetic dipole moment orientated along the z-axis and the electric field in the xz-plane, $\vec{m}$ and $\vec{E}_0$ can be expressed as:

$$\vec{m} = m\hat{e}_z \tag{3.21}$$

$$\vec{E}_0 = E_0\left(\cos(\psi)\hat{e}_z + \sin(\psi)\hat{e}_x\right) \tag{3.22}$$

where ψ is the angle between $\vec{E}_0$ and the z-axis (see Figure 3.2). Eqns. (3.20) and (3.22) give rise to the following expressions:

$$\mathcal{F}_M = \frac{1}{2}\left[E_0^2 - \left(\frac{c\mu_0}{4\pi}\right)^2\left\{\frac{3(\vec{m}\cdot\hat{e}_r)^2 + m^2}{r^6}\Theta(r-R) + \frac{4m^2}{R^6}\Theta(R-r)\right\}\right] \tag{3.23}$$

$$\mathcal{G}_M = \frac{c\mu_0}{4\pi}\left\{\frac{3(\vec{m}\cdot\hat{e}_r)(\vec{E}_0\cdot\hat{e}_r - (\vec{m}\cdot\vec{E}_0))}{r^3}\Theta(r-R) + \frac{2(\vec{m}\cdot\vec{E}_0)}{R^3}\Theta(R-r)\right\} \tag{3.24}$$

$$\vec{D}_M = \epsilon_0\vec{E}_0 \tag{3.25}$$

$$\vec{H}_M = \frac{1}{4\pi}\left[\frac{3(\vec{m}\cdot\hat{e}_r)\hat{e}_r - \vec{m}}{r^3}\Theta(r-R) + \frac{2\vec{m}}{R^3}\Theta(R-r)\right] \tag{3.26}$$

3.2.2 Calculating $\vec{\mathcal{E}}(\vec{x})$

Substituting eqns. (3.23), (3.24), (3.25) and (3.26) into eqn. (3.7) gives:

$$\begin{aligned}\vec{V}_1 =& 2E_0^2\epsilon_0\vec{E}_0 - \frac{\mu_0}{(4\pi)^2}\left\{6\frac{(\vec{m}\cdot\hat{e}_r)^2}{r^6}\vec{E}_0\Theta(r-R) + 2\frac{m^2\vec{E}}{r^6}\Theta(r-R) + 8\frac{m^2}{R^6}\vec{E}_0\Theta(R-r)\right.\\ & - 63\frac{(\vec{m}\cdot\hat{e}_r)^2(\vec{E}_0\cdot\hat{e}_r)\hat{e}_r}{r^6}\Theta(r-R) + 21\frac{(\vec{m}\cdot\hat{e}_r)(\vec{m}\cdot\vec{E})\hat{e}_r}{r^6}\Theta(r-R)\\ & \left. +21\frac{(\vec{m}\cdot\hat{e}_r)(\vec{E}_0\cdot\hat{e}_r)\vec{m}}{r^6}\Theta(r-R) - 7\frac{(\vec{m}\cdot\vec{E}_0)\vec{m}}{r^6}\Theta(r-R) - 28\frac{(\vec{m}\cdot\vec{E}_0)\vec{m}}{R^6}\Theta(r-R)\right\}\end{aligned}$$

Performing the divergence of $\vec{V}_1$ produces:

$$\begin{aligned}
\vec{\nabla}\cdot\vec{V}_1 = &-\frac{\mu_0}{(4\pi)^2}\left\{6\vec{\nabla}\cdot\left[\frac{(\vec{m}\cdot\hat{e}_r)^2}{r^6}\vec{E}_0\Theta(r-R)\right]+2\vec{\nabla}\cdot\left[\frac{m^2\vec{E}}{r^6}\Theta(r-R)\right]+8\vec{\nabla}\cdot\left[\frac{m^2}{R^6}\vec{E}_0\Theta(R-r)\right]\right.\\
&-63\vec{\nabla}\cdot\left[\frac{(\vec{m}\cdot\hat{e}_r)^2(\vec{E}_0\cdot\hat{e}_r)\hat{e}_r}{r^6}\Theta(r-R)\right]+21\vec{\nabla}\cdot\left[\frac{(\vec{m}\cdot\hat{e}_r)(\vec{m}\cdot\vec{E})\hat{e}_r}{r^6}\Theta(r-R)\right]\\
&+21\vec{\nabla}\cdot\left[\frac{(\vec{m}\cdot\hat{e}_r)(\vec{E}_0\cdot\hat{e}_r)\vec{m}}{r^6}\Theta(r-R)\right]-7\vec{\nabla}\cdot\left[\frac{(\vec{m}\cdot\vec{E}_0)\vec{m}}{r^6}\Theta(r-R)\right]\\
&\left.-28\vec{\nabla}\cdot\left[\frac{(\vec{m}\cdot\vec{E}_0)\vec{m}}{R^6}\Theta(r-R)\right]\right\}\\
=&\frac{\mu_0}{(4\pi)^2}\left\{9\frac{(\vec{m}\cdot\hat{e}_r)(\vec{m}\cdot\vec{E}_0)}{r^7}\Theta(r-R)-36\frac{(\vec{m}\cdot\hat{e}_r)^2(\vec{E}_0\cdot\hat{e}_r)}{r^7}\Theta(r-R)-9\frac{m^2(\vec{E}_0\cdot\hat{e}_r)}{r^7}\Theta(r-R)\right.\\
&\left.-42\frac{(\vec{m}\cdot\hat{e}_r)(\vec{m}\cdot\vec{E}_0)}{r^6}\delta(r-R)+36\frac{(\vec{m}\cdot\hat{e}_r)^2(\vec{E}_0\cdot\hat{e}_r)}{r^6}\delta(r-R)+6\frac{m^2(\vec{E}_0\cdot\hat{e}_r)}{r^6}\delta(r-R)\right\}
\end{aligned} \tag{3.27}$$

From the definition of $\vec{m}$ (eqn. 3.21) and $\vec{E}_0$ (eqn. 3.22), the following identities can be obtained:

$$\vec{m}\cdot\hat{e}_r = m\cos(\theta) \tag{3.28}$$

$$\vec{m}\cdot\vec{E}_0 = mE_0\cos(\psi) \tag{3.29}$$

$$\vec{E}_0\cdot\hat{e}_r = E_0\left[\sin(\psi)\sin(\theta)\cos(\phi)+\cos(\psi)\cos(\theta)\right] \tag{3.30}$$

Using the table of spherical harmonics [10], it is possible to show that:

$$\cos^2(\theta)\sin(\theta)\cos(\phi) = \frac{1}{5}\frac{1}{2}\left[\sqrt{\frac{16\pi}{21}}\left\{-Y_{31}+Y_{3-1}\right\}+\sqrt{\frac{8\pi}{3}}\left\{-Y_{11}+Y_{1-1}\right\}\right] \tag{3.31}$$

$$\cos^3(\theta) = \frac{1}{5}\left[\sqrt{\frac{16\pi}{7}}Y_{30}+3\sqrt{\frac{4\pi}{3}}Y_{10}\right] \tag{3.32}$$

$$\sin(\theta)\cos(\phi) = \frac{1}{2}\sqrt{\frac{8\pi}{3}}\left\{-Y_{11}+Y_{1-1}\right\} \tag{3.33}$$

$$\cos(\theta) = \sqrt{\frac{4\pi}{3}}Y_{10} \tag{3.34}$$

Using the identities (3.28)-(3.30) on eqn. (3.27) then converting to spherical harmonics gives:

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{V}_1 =& \frac{\mu_0 m^2 E_0}{(4\pi)^2}\Bigg\{9\frac{\cos(\psi)\cos(\theta)}{r^7}\Theta(r-R) - 36\frac{\cos^2(\theta)(\sin(\psi)\sin(\theta)\cos(\phi)+\cos(\psi)\cos(\theta))}{r^7}\Theta(r-R)\\
&- 9\frac{\sin(\psi)\sin(\theta)\cos(\phi)+\cos(\psi)\cos(\theta)}{r^7}\Theta(r-R) - 42\frac{\cos(\psi)\cos(\theta)}{r^6}\delta(r-R)\\
&+ 36\frac{\cos^2(\theta)(\sin(\psi)\sin(\theta)\cos(\phi)+\cos(\psi)\cos(\theta))}{r^6}\delta(r-R)\\
&+6\frac{\sin(\psi)\sin(\theta)\cos(\phi)+\cos(\psi)\cos(\theta)}{r^6}\delta(r-R)\Bigg\}\\
=& \frac{\mu_0 m^2 E_0}{(4\pi)^2}\Bigg\{\frac{9}{r^7}\Bigg[-\sin(\psi)\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left\{-Y_{11}+Y_{1-1}\right\} - 4\cos(\psi)\frac{1}{5}\left[\sqrt{\frac{16\pi}{7}}Y_{30}+3\sqrt{\frac{4\pi}{3}}Y_{10}\right]\\
&-4\sin(\psi)\frac{1}{5}\frac{1}{2}\left[\sqrt{\frac{64\pi}{21}}\left\{-Y_{31}+Y_{3-1}\right\}+\sqrt{\frac{8\pi}{3}}\left\{-Y_{11}+Y_{1-1}\right\}\right]\Bigg]\Theta(r-R)\\
&+\frac{6}{R^6}\Bigg[\cos(\psi)\sqrt{\frac{4\pi}{3}}Y_{10}+\sin(\psi)\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left\{-Y_{11}+Y_{1-1}\right\}\\
&+6\cos(\psi)\frac{1}{5}\left[\sqrt{\frac{16\pi}{7}}Y_{30}+3\sqrt{\frac{4\pi}{3}}Y_{10}\right]+\frac{6}{5}\sin(\psi)\frac{1}{2}\Bigg[\sqrt{\frac{64\pi}{21}}\left\{-Y_{31}+Y_{3-1}\right\}\\
&+\sqrt{\frac{8\pi}{3}}\left\{-Y_{11}+Y_{1-1}\right\}\Bigg]-7\cos(\psi)\sqrt{\frac{4\pi}{3}}Y_{10}\Bigg]\delta(r-R)\Bigg\}
\end{aligned}
$$

Computing eqn. (3.14) and then applying the orthogonality relation of spherical harmonics (eqn. (3.15)) gives:

$$
\begin{aligned}
\vec{\mathcal{E}}(\vec{x}) =& \frac{2\zeta\mu_0 m^2 E_0}{(4\pi)^3\epsilon_0^2}\vec{\nabla}_x\sum_{l,m}\frac{4\pi}{2l+1}Y_{lm}\int_0^\infty \mathrm{d}r'\; r'^2\frac{r_<^l}{r_>^{l+1}}\Bigg\{\frac{9}{r^7}\Bigg[-\sin(\psi)\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left\{-\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right\}\\
&-4\cos(\psi)\frac{1}{5}\left[\sqrt{\frac{16\pi}{7}}\delta_{l3}\delta_{m0}+3\sqrt{\frac{4\pi}{3}}\delta_{l1}\delta_{m0}\right]\\
&-4\sin(\psi)\frac{1}{5}\frac{1}{2}\left[\sqrt{\frac{64\pi}{21}}\left\{-\delta_{l3}\delta_{m1}+\delta_{l3}\delta_{m-1}\right\}+\sqrt{\frac{8\pi}{3}}\left\{-\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right\}\right]\Bigg]\Theta(r-R)\\
&+\frac{6}{R^6}\Bigg[\cos(\psi)\sqrt{\frac{4\pi}{3}}\delta_{l1}\delta_{m0}+\sin(\psi)\frac{1}{2}\sqrt{\frac{8\pi}{3}}\left\{-\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right\}\\
&+6\cos(\psi)\frac{1}{5}\left[\sqrt{\frac{16\pi}{7}}\delta_{l3}\delta_{m0}+3\sqrt{\frac{4\pi}{3}}\delta_{l1}\delta_{m0}\right]+\frac{6}{5}\sin(\psi)\frac{1}{2}\Bigg[\sqrt{\frac{64\pi}{21}}\left\{-\delta_{l3}\delta_{m1}+\delta_{l3}\delta_{m-1}\right\}\\
&+\sqrt{\frac{8\pi}{3}}\left\{-\delta_{l1}\delta_{m1}+\delta_{l1}\delta_{m-1}\right\}\Bigg]-7\cos(\psi)\sqrt{\frac{4\pi}{3}}\delta_{l1}\delta_{m0}\Bigg]\delta(r-R)\Bigg\}
\end{aligned}
$$

Computing the integration and then simplifying produces the final form of the induced electric field $\vec{\mathcal{E}}(\vec{x})$ of the spherical magnetic dipole:

$$\vec{\mathcal{E}}(\vec{x}) = -\vec{\nabla}_x \left\{ \frac{1}{4\pi\epsilon_0} \frac{\zeta\mu_0 m^2 E_0}{\pi\epsilon_0} \left[\frac{18}{5}\cos(\psi)\cos(\theta)\frac{1}{R^3 r^2} - \frac{9}{4}\cos(\psi)\cos^3(\theta)\frac{1}{r^5} + \frac{3}{10}\cos(\psi)\cos(\theta)\frac{1}{r^5} \right.\right. \tag{3.35}$$

$$\left.\left. -\frac{13}{10}\sin(\psi)\sin(\theta)\cos(\phi)\frac{1}{R^3 r^2} - \frac{9}{4}\sin(\psi)\sin(\theta)\cos^2(\theta)\cos(\phi)\frac{1}{r^5} \right] \right\} \tag{3.36}$$

From the expression for the induced electric field (eqn. (3.36)), we can see that $\vec{\mathcal{E}}(\vec{x})$ has a complicated angular dependence. By only considering the terms of the order $\frac{1}{r^2}$, we find that

$$\begin{aligned}
\vec{\mathcal{E}}(\vec{x}) &= -\vec{\nabla} \left\{ \frac{1}{4\pi\epsilon_0} \frac{\zeta\mu_0 m^2 E_0}{\pi\epsilon_0} \left[\frac{18}{5}\cos(\psi)\cos(\theta) - \frac{13}{5}\sin(\psi)\sin(\theta)\cos(\phi) \right] \frac{1}{R^3 r^2} \right\} \\
&= -\vec{\nabla} \left\{ \frac{1}{4\pi\epsilon_0} \frac{\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3} \frac{1}{r^2} \left[36(\cos(\psi)\cos(\theta) + \sin(\psi)\sin(\theta)\sin(\phi)) \right.\right. \\
&\quad \left.\left. -49\sin(\psi)\sin(\theta)\cos(\phi) \right] \right\} \\
&= -\vec{\nabla} \left\{ \frac{1}{4\pi\epsilon_0} \frac{\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3} \frac{1}{r^2} \left[36\frac{\vec{E}_0 \cdot \hat{e}_r}{E_0} - 49\frac{(\vec{E}_0 \cdot \hat{e}_x)(\hat{e}_x \cdot \hat{e}_r)}{E_0} \right] \right\}
\end{aligned} \tag{3.37}$$

From eqn. (3.37), we can associate the the electric dipole moment $\vec{p}_\psi$ with:

$$\vec{p}_\psi = \frac{\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3} \left[36\frac{\vec{E}_0}{E_0} - 49\frac{(\vec{E}_0 \cdot \hat{e}_x)\hat{e}_x}{E_0} \right] \tag{3.38}$$

such that $\vec{\mathcal{E}}(\vec{x}) = -\vec{\nabla}\left[\frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\vec{p}_\psi \cdot \hat{e}_r\right]$. If we now set $\vec{E}_0 || \vec{m}$ then $\vec{p}_\psi$ reduces to

$$\vec{p} = \frac{36\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\hat{e}_z \tag{3.39}$$

3.2.3 Calculating $\vec{\mathcal{B}}(\vec{x})$

Considering that there is already a classical magnetic field $\vec{B}_M$ which exists, it is expected that the induced magnetic field $\vec{\mathcal{B}}$, which is of the order ζ (see eqn. (2.19)), may be too small to be observable. Using the vector identity

$$\vec{\nabla} \times \left[f\vec{A}\right] = f\left[\nabla \times \vec{A}\right] + [\nabla f] \times \vec{A}$$

together with eqns. (3.23), (3.24), (3.25) and (3.26); an expression for $\vec{V}_2$ (eqn. (3.8)) can be obtained:

$$\begin{aligned}
\vec{\nabla} \times \vec{V}_2 =& \vec{\nabla} \times \left\{ -\frac{4}{c}\mathcal{F}_M \vec{H}_M + 7\mathcal{G}_M \vec{D}_M \right\} \\
=& -\frac{4}{c}\mathcal{G}_M \left[\vec{\nabla} \times \vec{H}_M \right] - \frac{4}{c}\left[\vec{\nabla}\mathcal{G}_M \right] \times \vec{H}_M + 7\left[\vec{\nabla}\mathcal{F}_M \right] \times \vec{D}_M \\
=& \frac{1}{4\pi c}\left\{ 6E_0^2 \frac{\hat{e}_r \times \vec{m}}{r^3}\delta(r-R) + \left(\frac{\mu_0 c}{4\pi}\right)^2 \left[12\frac{(\vec{m}\cdot\hat{e}_r)^2(\hat{e}_r\times\vec{m})}{r^{10}}\Theta(r-R) \right.\right. \\
&+ 12\frac{m^2(\hat{e}_r\times\vec{m})}{r^{10}}\Theta(r-R) - 6\frac{(\vec{m}\cdot\hat{e}_r)^2(\hat{e}_r\times\vec{m})}{r^9}\delta(r-R) \\
&\left. -18\frac{m^2(\hat{e}_r\times\vec{m})}{r^9}\delta(r-R) \right] + 21\frac{(\vec{m}\times\vec{E}_0)(\vec{E}_0\cdot\hat{e}_r)}{r^4}\Theta(r-R) \\
&- 105\frac{(\vec{m}\cdot\hat{e}_r)(\vec{E}_0\cdot\hat{e}_r)(\hat{e}_r\times\vec{E}_0)}{r^4}\Theta(r-R) + 21\frac{(\vec{m}\cdot\vec{E}_0)(\hat{e}_r\times\vec{E}_0)}{r^4}\Theta(r-R) \\
&\left. +21\frac{(\vec{m}\cdot\hat{e}_r)(\vec{E}_0\cdot\hat{e}_r)(\hat{e}_r\times\vec{E}_0)}{r^3}\delta(r-R) - 21\frac{(\vec{m}\cdot\vec{E}_0)(\hat{e}_r\times\vec{E}_0)}{r^3}\delta(r-R) \right\}
\end{aligned} \tag{3.40}$$

Using the definitions of $\vec{m}$ and $\vec{E}_0$, the following identities can be derived:

$$\begin{aligned}
\vec{m} \times \vec{E}_0 &= mE_0 \sin(\psi)\hat{e}_y \\
\hat{e}_r \times \vec{E}_0 &= E_0\left[\cos(\psi)\sin(\theta)\sin(\phi)\hat{e}_x + (\sin(\psi)\cos(\theta) - \cos(\psi)\sin(\theta)\cos(\phi))\,\hat{e}_y - \sin(\psi)\sin(\theta)\sin(\phi)\hat{e}_z\right] \\
\hat{e}_r \times \vec{m} &= m\sin(\theta)\left[\sin(\phi)\hat{e}_x - \cos(\phi)\hat{e}_y\right]
\end{aligned}$$

All that is left is to perform the integration (eqn. (3.9)). To get a feel for the magnitude of $\vec{\mathcal{B}}$, we consider the following terms:

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|}\frac{\hat{e}'_r \times \vec{m}}{r'^3}\delta(r'-R) = \frac{4\pi}{3}\frac{1}{r^2}\hat{e}_r \times \vec{m} \tag{3.41}$$

$$\begin{aligned}
\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|}\frac{(\vec{m}\cdot\hat{e}'_r)^2(\hat{e}'_r\times\vec{m})}{r'^9}\delta(r'-R) =& m^2\left\{ \frac{4\pi}{7}\frac{1}{R^4r^4}\frac{1}{5}(5\cos^2(\theta)-1)\hat{e}_r\times\vec{m} \right. \\
&\left. + \frac{4\pi}{3}\frac{1}{R^6r^2}\frac{1}{5}\hat{e}_r\times\vec{m} \right\}
\end{aligned} \tag{3.42}$$

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|}\frac{m^2(\hat{e}'_r\times\vec{m})}{r'^9}\delta(r'-R) = m^2\frac{4\pi}{3}\frac{1}{R^6r^2}\hat{e}_r\times\vec{m} \tag{3.43}$$

$$\begin{aligned}
\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|}\frac{(\vec{m}\cdot\hat{e}'_r)^2(\hat{e}'_r\times\vec{m})}{r'^{10}}\Theta(r'-R) =& m^2\left[\frac{4\pi}{7}\left\{ \frac{7}{44}\frac{1}{r^8} + \frac{1}{4}\frac{1}{R^4r^4} \right\}\frac{1}{5}(5\cos^2(\theta)-1)\hat{e}_r\times\vec{m} \right. \\
&\left. + \frac{4\pi}{3}\left\{ \frac{1}{6}\frac{1}{R^6r^2} - \frac{1}{18}\frac{1}{r^8} \right\}\frac{1}{5}\hat{e}_r\times\vec{m} \right]
\end{aligned} \tag{3.44}$$

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{m^2(\hat{e}'_r \times \vec{m})}{r'^{10}} \theta(r'-R) = m^2 \frac{4\pi}{3} \left\{ \frac{1}{6} \frac{1}{R^6 r^2} - \frac{1}{18} \frac{1}{r^8} \right\} \hat{e}_r \times \vec{m} \quad (3.45)$$

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{(\vec{E}_0 \cdot \hat{e}'_r)(\vec{m} \times \vec{E})}{r'^4} \Theta(r'-R) = m E_0^2 \frac{4\pi}{3} \left\{ \frac{1}{r^2} \ln \frac{r}{R} + \frac{1}{3} \frac{1}{r^2} \right\} (\vec{E}_0 \cdot \hat{e}_r)(\vec{m} \times \vec{E}_0) \quad (3.46)$$

$$\begin{aligned}
\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{(\vec{m} \cdot \hat{e}'_r)(\vec{E}_0 \cdot \hat{e}'_r)(\hat{e}'_r \times \vec{E}_0)}{r'^3} \delta(r'-R) &= m E_0^2 \Bigg\{ \frac{4\pi}{7} \frac{R^2}{r^4} \\
&\Bigg[\frac{1}{2} \sin(\psi)\cos(\psi)\sin^2(\theta)\cos(\theta)\sin(2\phi)\hat{e}_x \\
&+ \frac{1}{5}\cos^2(\psi)\sin(\theta)(5\cos^2(\theta)-1)\sin(\phi)\hat{e}_x - \frac{1}{5}\cos(2\psi)\sin(\theta)(5\cos^3(\theta)-1)\cos(\phi)\hat{e}_y \\
&+ \frac{3}{10}\cos(\psi)\sin(\psi)(5\cos^2(\theta)-3\cos(\theta))\hat{e}_y - \frac{1}{2}\cos(\psi)\sin(\psi)\sin^2(\theta)\cos(\theta)\cos(2\phi)\hat{e}_y \\
&- \frac{1}{2}\sin^2(\psi)\sin^2(\theta)\cos(\theta)\sin(2\phi)\hat{e}_z - \frac{1}{5}\cos(\psi)\sin(\psi)\sin(\theta)(5\cos^2(\theta)-1)\sin(\phi)\hat{e}_z \Bigg] \\
&+ \frac{4\pi}{3}\frac{1}{r^2} \Bigg[\frac{1}{5}\cos^2(\psi)\sin(\theta)\sin(\phi)\hat{e}_x - \frac{1}{5}\cos(2\psi)\sin(\theta)\cos(\phi)\hat{e}_y + \frac{4}{10}\cos(\psi)\sin(\psi)\cos(\theta)\hat{e}_y \\
&- \frac{1}{5}\cos(\psi)\sin(\psi)\sin(\theta)\sin(\phi)\hat{e}_z \Bigg] \Bigg\} \quad (3.47)
\end{aligned}$$

$$\begin{aligned}
\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{(\vec{m} \cdot \hat{e}'_r)(\vec{E}_0 \cdot \hat{e}'_r)(\hat{e}'_r \times \vec{E}_0)}{r'^4} \theta(r'-R) &= m E_0^2 \Bigg\{ \frac{4\pi}{7} \Bigg\{ \frac{7}{10} \frac{1}{r^2} \\
&- \frac{1}{2} \frac{R^2}{r^4} \Bigg\} \Bigg[\frac{1}{2} \sin(\psi)\cos(\psi)\sin^2(\theta)\cos(\theta)\sin(2\phi)\hat{e}_x \\
&+ \frac{1}{5}\cos^2(\psi)\sin(\theta)(5\cos^2(\theta)-1)\sin(\phi)\hat{e}_x - \frac{1}{5}\cos(2\psi)\sin(\theta)(5\cos^3(\theta)-1)\cos(\phi)\hat{e}_y \\
&+ \frac{3}{10}\cos(\psi)\sin(\psi)(5\cos^2(\theta)-3\cos(\theta))\hat{e}_y - \frac{1}{2}\cos(\psi)\sin(\psi)\sin^2(\theta)\cos(\theta)\cos(2\phi)\hat{e}_y \\
&- \frac{1}{2}\sin^2(\psi)\sin^2(\theta)\cos(\theta)\sin(2\phi)\hat{e}_z - \frac{1}{5}\cos(\psi)\sin(\psi)\sin(\theta)(5\cos^2(\theta)-1)\sin(\phi)\hat{e}_z \Bigg] \\
&+ \frac{4\pi}{3}\frac{1}{r^2}\left(\ln\frac{r}{R} + \frac{1}{3}\right) \Bigg[\frac{1}{5}\cos^2(\psi)\sin(\theta)\sin(\phi)\hat{e}_x - \frac{1}{5}\cos(2\psi)\sin(\theta)\cos(\phi)\hat{e}_y \\
&+ \frac{4}{10}\cos(\psi)\sin(\psi)\cos(\theta)\hat{e}_y - \frac{1}{5}\cos(\psi)\sin(\psi)\sin(\theta)\sin(\phi)\hat{e}_z \Bigg] \Bigg\} \quad (3.48)
\end{aligned}$$

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{(\vec{m} \cdot \vec{E}_0)(\hat{e}'_r \times \vec{E}_0)}{r'^3} \delta(r'-R) = \frac{4\pi}{3} \frac{(\vec{m} \cdot \vec{E}_0)(\hat{e}_r \times \vec{E}_0)}{r^2} \quad (3.49)$$

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{(\vec{m} \cdot \vec{E}_0)(\hat{e}'_r \times \vec{E}_0)}{r'^4} \theta(r'-R) = \frac{4\pi}{3} \frac{1}{r^2} \left(\ln\frac{r}{R} + \frac{1}{3}\right) (\vec{m} \cdot \vec{E}_0)(\hat{e}_r \times \vec{E}_0) \quad (3.50)$$

The terms (3.47) and (3.48) have a complicated angular dependence. To make things easier, we choose $\vec{E}_0||\vec{m}$. This results in the angle $\psi = 0$. Term (3.46) becomes 0 since $\vec{m} \times \vec{E}_0 = 0$. The angular dependence of terms (3.47) and (3.48) also becomes less complicated and reduce to:

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{(\vec{m}\cdot\hat{e}'_r)(\vec{E}_0\cdot\hat{e}'_r)(\hat{e}'_r\times\vec{E}_0)}{r'^3}\delta(r'-R) = E_0^2\frac{1}{5}\left\{\frac{4\pi}{7}\frac{R^2}{r^4}\left[5\cos^2(\theta)-1\right] + \frac{4\pi}{3}\frac{1}{r^2}\right\}(\hat{e}_r\times\vec{m}) \tag{3.51}$$

$$\int \mathrm{d}^3y \frac{1}{|\vec{x}-\vec{y}|} \frac{(\vec{m}\cdot\hat{e}'_r)(\vec{E}_0\cdot\hat{e}'_r)(\hat{e}'_r\times\vec{E}_0)}{r'^4}\theta(r'-R) = \frac{1}{5}E_0^2\left\{\frac{4\pi}{7}\left\{\frac{7}{10}\frac{1}{r^2} - \frac{1}{2}\frac{R^2}{r^4}\right\}\left[5\cos^2(\theta)-1\right] + \frac{4\pi}{3}\frac{1}{r^2}\left(\ln\frac{r}{R}+\frac{1}{3}\right)\right\}(\hat{e}_r\times\vec{m}) \tag{3.52}$$

The $\ln\frac{r}{R}$ dependence of the terms (3.52) and (3.50) is of concern. If one takes into account the numeric prefactors of the terms (eqn. (3.40)) then the $\ln\frac{r}{R}$ terms cancel each other out. After taking all of this into account, the induced magnetic field $\vec{\mathcal{B}}(\vec{x})$ can be expressed as,

$$\vec{\mathcal{B}}(\vec{x}) = \vec{\nabla}\times\left\{\frac{2\zeta}{4\pi\epsilon_0^2c^2}\left[E_0^2\left\{-\frac{57}{10}\frac{1}{r^2}+\frac{21}{10}\frac{R^2}{r^4}\right\} + m^2\left(\frac{\mu_0 c}{4\pi}\right)^2\left\{\frac{1}{r^8}\left[\frac{3}{55}(5\cos^2(\theta)-1)-\frac{12}{45}\right] - \frac{3}{35}\frac{1}{R^4r^4}(5\cos^2-1) - \frac{84}{15}\frac{1}{R^6r^2}\right\}\right]\right\}\hat{e}_r\times\vec{m} \tag{3.53}$$

We define,

$$\vec{A}_1 = -\frac{24}{35}\frac{\zeta m^2\mu_0^2}{(4\pi)^3\epsilon_0^2}\frac{1}{R^4r^4}\hat{e}_r\times\vec{m} \qquad \vec{A}_2 = -\frac{168}{15}\frac{\zeta m^2\mu_0^2}{(4\pi)^3\epsilon_0^2}\frac{1}{R^6r^2}\hat{e}_r\times\vec{m}$$

$$\vec{A}_3 = -\frac{48}{495}\frac{\zeta m^2\mu_0^2}{(4\pi)^3\epsilon_0^2}\frac{1}{r^8}\hat{e}_r\times\vec{m}$$

$$\vec{A}_4 = -\frac{114}{10}\frac{\zeta E_0^2\mu_0}{4\pi\epsilon_0}\frac{1}{r^2}\hat{e}_r\times\vec{m} \qquad \vec{A}_5 = \frac{42}{10}\frac{\zeta E_0^2\mu_0}{4\pi\epsilon_0}\frac{R^2}{r^4}\hat{e}_r\times\vec{m}$$

where A_1 and A_3 represent the maximum value of the angular dependent terms in eqn. (3.53). The magnetic field $\vec{B}_d$ produced by a magnetic dipole moment $\vec{m} = m\hat{e}_z$ is given by $\vec{B}_d = \vec{\nabla}\times\vec{A}$ where the vector potential $\vec{A}$ is given by:

$$\vec{A} = -\frac{\mu_0}{4\pi}\frac{1}{r^2}\hat{e}_r\times\vec{m}$$

Using SI Units, $\zeta \approx 10^{-52}$, $\epsilon_0 \approx 10^{-11}$ and $\mu_0 \approx 10^{-6}$. We can now compare the relative strengths of the $\vec{A}_i$ terms

$$\frac{|\vec{A}_1|}{|\vec{A}|} \approx 4 \times 10^{-39} \frac{m^2}{R^4 r^2}$$
$$\frac{|\vec{A}_2|}{|\vec{A}|} \approx 7 \times 10^{-38} \frac{m^2}{R^6}$$
$$\frac{|\vec{A}_3|}{|\vec{A}|} \approx 6 \times 10^{-40} \frac{m^2}{r^6}$$
$$\frac{|\vec{A}_4|}{|\vec{A}|} \approx 10^{-40} E_0^2$$
$$\frac{|\vec{A}_5|}{|\vec{A}|} \approx 4 \times 10^{-41} \frac{E_0^2 R^2}{r^2}$$

With reasonable choices for the magnetic moment, m, and the radius of the spherical shell, R; the strength of the induced magnetic field will be much smaller than the magnetic field produced by the the magnetic moment, m.

3.2.4 Summary and Discussion

Similarly to the charged spherical shell, there arises a restriction to r from the constraint $|c\vec{B}_M| < 1.3 \times 10^{18}$. Using the maximum value for $\vec{B}_M$, namely $\theta = 0$, and $r > R$,

$$|c\vec{B}_M| = \frac{\mu_0 c}{2\pi} \frac{m}{r^3} < 1.3 \times 10^{18}$$
$$r^3 > \frac{1}{1.3 \times 10^{18}} \frac{\mu_0 cm}{2\pi}$$

Thus,

$$r > 3.59 \times 10^{-6} \sqrt[3]{m} \tag{3.54}$$

where r is measured in meters and m is measured in Ampere meter^2.

In the case of the neutron, $r > 3.59 \times 10^{-6} \sqrt[3]{10^{-26}} \approx 7.7 \times 10^{-15}$ which is greater than the radius of the neutron, $R_n \approx 0.4 \times 10^{-15}$. This means that the induced fields cannot be evaluated close to the neutron since the weak field expansion of the Heisenberg-Euler Lagrangian will not be valid in that region.

The leading term in the induced magnetic field is the dipole term, $\frac{1}{r^2}$. However, this term is very small compared to the magnetic field produced by $\vec{m}$ and can therefore be safely ignored.

The induced electric field on the other hand produces an electric dipole seen in eqn. (3.38).

$$\vec{p}_\psi = \frac{\zeta \mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\left[36\frac{\vec{E}_0}{E_0} - 49\frac{(\vec{E}_0 \cdot \hat{e}_x)\hat{e}_x}{E_0}\right]$$

This electric dipole has a very strange dependence on the angle ψ. It must be noted that if $\psi = 0$, which is equivalent to $\vec{E}_0 || \vec{m}$, then eqn. (3.38) reduces to eqn. (3.39):

$$\vec{p} = \frac{36\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\hat{e}_z$$

Chapter 4

Experimental Observability and Conclusion

4.1 Experimental Observability

We see from the preceding chapter that the Heisenberg-Euler Lagrangian predicts some interesting effects, namely the presence of a magnetic dipole induced by the interaction between a uniformly charged spherical shell and an external quasi-static magnetic field; as well as the presence of an electric dipole induced by the interaction between a spherical magnetic dipole and an external quasi-static electric field.

These induced fields are proportional to the constant $\zeta = \frac{2\alpha^2\epsilon_0^2\hbar}{45m^4c^5} \approx 1.3 \times 10^{-52}\frac{Jm}{V^4}$ and are therefore very small. However, these effects are not predicted by the usual Maxwell's theory and should therefore be detectable when using appropriate external fields and sources. Various experimental setups are being considered for possible observations of these induced effects. Some of these setups will be discussed now.

4.1.1 Charged Spherical Shell in a Quasi-static Magnetic Field

The easiest object that one could use as a charged spherical shell would be the proton. However as discussed in Chapter 3, the radius of the proton is less than the restriction to the radial distance from the centre of the spherical shell of charge $Q = 1.9 \times 10^{-19}C$. As a consequence of this, the measurement of the magnetic dipole moment cannot be taken at distances closer than $20 \times 10^{-15}m$.

If we consider a charged spherical shell such that the electric field strength at the radius of the shell is equivalent to the electric field strength required to cause dielectric breakdown in air, $|\vec{E}_d| = 3 \times 10^6 V/m$, then,

$$|\vec{E}_d| = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^2} = 3 \times 10^6$$

$$\Rightarrow \frac{Q}{R^2} = 12\pi\epsilon_0 \times 10^6$$

Substituting this into the induced magnetic moment (eqn. (3.19)) leads to,

$$\begin{aligned} |\vec{m}| &= \frac{c^2\zeta}{6\pi\epsilon_0^2}\frac{Q^2}{R}|\vec{B}_0| \\ &= 24\pi c^2\zeta R^3|\vec{B}_0| \times 10^{12} \\ &\approx 8.8 \times 10^{-22}\ R^3|\vec{B}_0| \end{aligned}$$

The induced magnetic moment is measured in Ampere meter^2. If we consider an external quasi-static magnetic field of strength 1 Tesla then the induced magnetic moment is given by

$$|\vec{m}| \approx 10^{-21}R^3$$

Choosing a spherical shell of radius $1mm$ produces an induced magnetic dipole of magnitude $|\vec{m}| \approx 10^{-30}Am^2$. This magnetic dipole moment is very small but is comparable to that of magnetic moment of the proton $|\vec{m}_p| \approx 1.40869 \times 10^{-26}Am^2$. Increasing the radius of the spherical shell will increase the magnitude of the magnetic dipole but this too increases the charge Q required on the surface of the shell. Using a radius of $1mm$ requires the charge on the shell to be $0.3 \times 10^{-9}C$, which equates to 2×10^9 electrons. Creating a spherical shell with larger charges may prove challenging.

Note that the restriction to the radius of the spherical shell is of no concern since the electric field strength has been chosen such that it is less than the critical electric field strength, namely $|\vec{E}_d| = 3 \times 10^6 V/m < 1.3 \times 10^{18} V/m$.

4.1.2 Spherical Magnetic Dipole in a Quasi-static Electric Field

The spherical magnetic dipole in an electric field is considered the most likely configuration for the observation of the induced fields in non-linear quantum electrodynamics. A neutron would make the best candidate for the spherical dipole. However, it must be noted that radius of the neutron is smaller than the restriction to the radial distance, $r > 7.7 \times 10^{-15}$. This means that detector may not get too close to the neutron since the weak field approximation of the Heisenberg-Euler Lagrangian will not be valid when field strengths are greater than the critical field strength $\epsilon_c = 1.3 \times 10^{18}\ V/m$.

If one considers the case where the external electric field is aligned with the magnetic moment of the neutron, $\vec{E}_0||\vec{m}$, then

$$\begin{aligned} |\vec{p}| &= \frac{36\zeta\mu_0 m^2}{10\pi\epsilon_0 R^3}E_0 \\ &\approx 1.3 \times 10^{-33}E_0 \end{aligned}$$

where $|\vec{p}|$ is measured in $|e|\ cm$ and E_0 in V/m.

Current measurements of the electric dipole moment of the neutron are of the order $10^{-26}|e|\ cm$. In order to bring the induced dipole moment to such a level, one would require an electric field of the order $E_0 \approx 10^7\ V/m$.

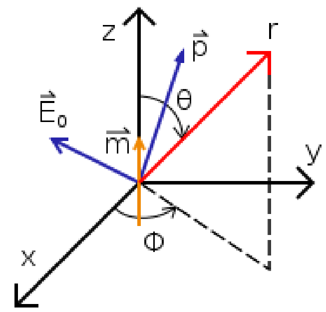

Figure 4.1: The induced electric dipole moment in relation to the external electric field

In the more general case where $\psi \neq 0$, we see that $\vec{p}_\psi$ has a strange angular dependence (eqns. (3.38) and (3.39)).

$$\vec{p}_\psi = \frac{\zeta \mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\left[36\frac{\vec{E}_0}{E_0} - 49\frac{(\vec{E}_0 \cdot \hat{e}_x)\hat{e}_x}{E_0}\right]$$

This strange angular dependence on the external electric field would help to distinguish between the induced electric dipole moment and the electric dipole moment of the neutron. Figure 4.1 shows this angular dependence, where the external electric field, $\vec{E}_0$, and the induced electric dipole moment, $\vec{p}_\psi$ lie in the $x - z$ plane. The position vector $\vec{r}$ is also shown where r is the radial displacement, θ is the polar angle and ϕ is the azimuthal angle.

The neutron's permanent electric dipole moment arises from a violation of the CP-symmetry [2]. The electric dipole moment of the neutron has yet to be observed. However, the upper limit on the strength of the electric dipole moment had been measured and improved upon over the last sixty years. The first experiment to detect the neutron's electric dipole moment was completed in 1951 by J. Smith, E. Purcell and N. Ramsey. They found that the electric dipole moment of the neutron had to be less than $5 \times 10^{-20} e\ cm$ [13]. Recently published results by C. A. Baker et al. in Physical Review Letters of 2006, places the upper limit of the electric dipole moment of the neutron at $2.9 \times 10^{-26} e\ cm$ [1].

The experiments preformed to determine the upper limit of the neutron's electric dipole moment use the changes in the Larmor frequency of the neutron's magnetic moment to determine the upper limit [1]. The Larmor precession arises from the magnetic moment trying to align itself with the external magnetic field. The additional electric dipole moment would also want to align itself with the external electric field. This will either increase the Larmor frequency or decrease it depending on the alignment of the external fields. In order to detect the induced electric dipole moment of the neutron, one could follow the same technique. However, the experimental setup would not be same. As stated previously, the strength of the external electric field would need to be of the order $E_0 \approx 10^7\ V/m$ for the induced electric moment of the neutron to be comparable with the permanent electric dipole moment of the neutron. These types of electric field strengths are, however, found in some crystals. Letting the neutrons pass through one of these crystals would be sufficient to induce an electric dipole moment. By applying an external magnetic field, one should observe the induced electric dipole moment as a change in the Larmor frequency. The observability of the induced electric dipole moment is also helped by the strange angular dependence which would not be present in the electric dipole moment of the neutron.

Another possible experimental setup is the one discussed by V. Fedorov et al in the paper "Measurement of the neutron electric dipole moment by crystal diffraction" [6]. With this setup, neutrons passing through a crystal undergo a spin rotation. This rotation is then detected and used to determine the electric dipole moment of the neutron.

4.2 Outlook

Apart from the experiment to look for these induced electric and magnetic field, there are many more configurations to be considered. The only configurations that have been considered so far were the magnetic or electric sources in external quasi-static fields. Future possibilities could include both electric and magnetic sources in quasi-static fields. However, this will only produce more complicated expressions for $\vec{V}_1(\vec{x})$ (eqn. (3.7)) and $\vec{V}_2(\vec{x})$ (eqn. (3.8)).

The other possible adaption that could be made, is to derive $\vec{\mathcal{E}}(\vec{x})$ and $\vec{\mathcal{B}}(\vec{x})$ using time-dependant external fields. In other words, Maxwell's equations would take on the more general form:

$$\vec{\nabla} \cdot \vec{D} = j_0 \qquad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
$$\vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{H} = \vec{j} + \frac{\partial \vec{D}}{\partial t}$$

This would make $\vec{\mathcal{E}}(\vec{x})$ and $\vec{\mathcal{B}}(\vec{x})$ more general.

4.3 Conclusion

We have seen so far that the Heisenberg-Euler Lagrangian implies some interesting interaction between classical electric and magnetic field sources; and external quasi-static electric and magnetic fields [3]. This interaction leads to correction to the fields produces by the sources as well as new fields, in the case of pure electric and pure magnetic sources. These changes to the fields are represented by the induced fields $\vec{\mathcal{E}}(\vec{x})$ and $\vec{\mathcal{B}}(\vec{x})$:

$$\vec{\mathcal{E}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y \cdot \left\{8\mathcal{F}_M\vec{D}_M + \frac{14}{c}\mathcal{G}_M\vec{H}_M\right\}$$
$$\vec{\mathcal{B}}(\vec{x}) = \frac{\zeta}{4\pi\epsilon_0^2}\vec{\nabla}_x \int \frac{\mathrm{d}^3 y}{|\vec{x}-\vec{y}|}\vec{\nabla}_y \times \left\{-\frac{8}{c}\mathcal{F}_M\vec{H}_M + 14\mathcal{G}_M\vec{D}_M\right\}$$

If we consider a uniformly charged spherical shell as a source for an electric fields then we find that the external quasi-static magnetic field induces a magnetic dipole moment in the shell. The correction to the electric field produced by the charged spherical shell is found to be significantly smaller than the electric field and will therefore be unobservable when detecting the electric field strength. However, the induced magnetic dipole moment is something new and should be detectable for the right combination of the total surface charge Q, the radius of the spherical shell R and the strength of the external quasi-static magnetic field B_0. The magnetic dipole moment $\vec{m}$ is given by the expression:

$$\vec{m} = \frac{c^2\zeta}{6\pi\epsilon_0^2}\frac{Q^2}{R}\vec{B}$$

Using a charged spherical shell of total charge $Q = 0.3 \times 10^{-9}C$ and radius $R = 1mm$ in the presence of an external magnetic field of strength $B_0 = 1T$, produces a magnetic dipole moment of strength $|\vec{m}| \approx 10^{-30}Am^2$.

The second case that was considered was a spherical shell with a current $\vec{j} = \frac{3|\vec{m}|}{4\pi R^3}\delta(r-R)\hat{e}_\phi$ moving on the surface of the shell. For distances greater than the radius of the shell, R, the magnetic field produced by the current $\vec{j}$ is the same as the magnetic field produced by the magnetic dipole moment $\vec{m}$. An external quasi-static electric field is applied over the shell and leads to the production of the induced fields. The correction to the magnetic field produced by the current $\vec{j}$ is small compared to the magnetic field and can safely be ignored. The induced electric field has the same form as an electric dipole. However, this induced electric dipole moment has a strange dependence on the angle, ψ, between the magnetic dipole moment and the external quasi-static electric field. The electric dipole moment $\vec{p}_\psi$ is given by:

$$\vec{p}_\psi = \frac{\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\left[36\frac{\vec{E}_0}{E_0} - 49\frac{(\vec{E}_0\cdot\hat{e}_x)\hat{e}_x}{E_0}\right]$$

For the case where $\vec{E}_0||\vec{m}$, namely $\psi = 0$, $\vec{p}_\psi$ reduces to $\vec{p}$:

$$\vec{p} = \frac{36\zeta\mu_0 m^2 E_0}{10\pi\epsilon_0 R^3}\hat{e}_z$$

These results are very exciting and should produce some interesting experimental results. As electric and magnetic field strengths increase, so too does the observability of the induces electric and magnetic fields. Experimental setups are being devised to detect and measure these induced fields using present technology. The future of this subject looks very promising with possible experiments on the line as well as possible adaptions to the derivation of the induced fields $\vec{\mathcal{E}}(\vec{x})$ and $\vec{\mathcal{B}}(\vec{x})$.

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